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# An Introduction to Noncommutative Geometry

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# Introduction

This book consists of lecture notes for a course given at the EMS Summer School on Noncommutative Geometry and Applications, at Monsaraz and Lisboa, Portugal in September, 1997. These were made available in preprint form on the ArXiv, as physics/9709045, at that time. In updating them for publication, I have kept to the original plan, but have added citations of more recent papers throughout. An extra final chapter summarizes some of the developments in noncommutative geometry in the intervening years.

The course sought to address a mixed audience of students and young researchers, both mathematicians and physicists, and to provide a gateway to noncommutative geometry, as it then stood. It already occupied a wide-ranging area of mathematics, and had received some scrutiny from particle physicists. Shortly thereafter, links to string theory were found, and its interest for theoretical physicists is now indisputable.

Many approaches can be taken to introducing noncommutative geometry. In these lectures, the focus is on the geometry of Riemannian spin manifolds and their noncommutative cousins, which are 'spectral triples' determined by a suitable generalization of the Dirac operator. These 'spin geometries', which are spectral triples with certain extra properties, underlie the noncommutative geometry approach to phenomenological particle models and recent attempts to place gravity and matter fields on the same geometrical footing.

The first two chapters are devoted to commutative geometry; we set up the general framework and then compute a simple example, the two-sphere, in noncommutative terms. The general definition of a spin geometry is then laid out and exemplified with the noncommutative torus. Enough details are given so that one can see clearly that noncommutative geometry is just ordinary geometry, extended by discarding the commutativity assumption on the coordinate algebra. Classification up to equivalence is dealt with briefly in Chapter 7.

Other chapters explore some of the tools of the trade: the noncommutative integral, the role of quantization, and the spectral action functional. Physical models are not treated directly (these were the subject of other lectures at the Summer School), but most of the mathematical issues needed for their understanding are dealt with here. The final chapter is a brief overview of the profusion of new examples and applications of noncommutative spaces and spectral triples.

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# 1

# Commutative geometry from the noncommutative point of view

The traditional arena of geometry and topology is a *set of points* with some particular structure that, for want of a better name, we call a *space*. Thus, for instance, one studies curves and surfaces as subsets of an ambient Euclidean space. It was recognized early on, however, that even such a fundamental geometrical object as an elliptic curve is best studied not as a set of points (a torus) but rather by examining functions on this set, specifically the doubly periodic meromorphic functions. Weierstrass opened up a new approach to geometry by studying directly the collection of complex functions that satisfy an algebraic addition theorem, and derived the point set as a consequence. In probability theory, the set of outcomes of an experiment forms a measure space, and one may regard events as subsets of outcomes; but most of the information is obtained from 'random variables', i.e., measurable functions on the space of outcomes.

In noncommutative geometry, under the influence of quantum physics, this general idea of replacing sets of points by classes of functions is taken further. In many cases the set is completely determined by an algebra of functions, so one forgets about the set and obtains all information from the functions alone. Also, in many geometrical situations the associated set is very pathological, and a direct examination yields no useful information. The set of orbits of a group action, such as the rotation of a circle by multiples of an irrational angle, is of this type. In such cases, when we examine the matter from the algebraic point of view, we often obtain a perfectly good operator algebra that holds the information we need; however, this algebra is generally not commutative. Thus, we proceed by first discovering how function algebras determine the structure of point sets, and then learning which relevant properties of function algebras do not depend on commutativity.

In a famous paper [94] that has become a cornerstone of noncommutative geometry, Gelfand and Naĭmark in 1943 characterized the involutive algebras of operators by just dropping commutativity from the most natural axiomatization for the algebra of continuous functions on a locally compact Hausdorff space. The starting point for noncommutative geometry that we adopt here is to study ordinary 'commutative' spaces via their algebras of functions, omitting wherever possible any reference to the commutativity of these algebras.

## 1.1 The Gelfand–Naĭmark cofunctors

The Gelfand–Naĭmark theorem can be thought of as the construction of two contravariant functors (*cofunctors* for short) from the category of locally compact Hausdorff spaces to the category of $C^*$-algebras.

The first cofunctor $C$ takes a compact space $X$ to the $C^*$-algebra $C(X)$ of continuous complex-valued functions on $X$, and takes a continuous map $f\colon X \to Y$ to its transpose $Cf\colon C(Y) \to C(X)$, $h \mapsto h \circ f$. If $X$ is only a locally compact space, the corresponding $C^*$-algebra is $C_0(X)$ whose elements are continuous functions vanishing at infinity, and we require that the continuous maps $f\colon X \to Y$ be proper (the preimage of a compact set is compact) in order that $h \mapsto h \circ f$ take $C_0(Y)$ into $C_0(X)$.

The other cofunctor $M$ goes the other way: it takes a $C^*$-algebra $A$ onto its space of *characters*, that is, nonzero homomorphisms $\mu\colon A \to \mathbb{C}$. If $A$ is unital, $M(A)$ is closed in the weak* topology of the unit ball of the dual space $A^*$ and hence is compact. If $\phi\colon A \to B$ is a unital $*$-homomorphism, the cofunctor $M$ takes $\phi$ to its transpose $M\phi\colon M(B) \to M(A)$, $\mu \mapsto \mu \circ \phi$.

Write $X^+ := X \uplus \{\infty\}$ for the space $X$ with a point at infinity adjoined (whether $X$ is compact or not), and write $A^+ := \mathbb{C} \times A$ for the $C^*$-algebra $A$ with an identity adjoined via the rule $(\lambda, a)(\mu, b) := (\lambda\mu, \lambda b + \mu a + ab)$, whether $A$ is unital or not; then $C(X^+) \simeq C_0(X)^+$ as unital $C^*$-algebras. If $\mu_0\colon A^+ \to \mathbb{C}$, $(\lambda, a) \mapsto \lambda$, then $M(A) = M(A^+) \setminus \{\mu_0\}$ is locally compact when $A$ is nonunital. Notice that $M(A)^+$ and $M(A^+)$ are homeomorphic.

That no information is lost in passing from spaces to $C^*$-algebras can be seen as follows. If $x \in X$, the *evaluation* $f \mapsto f(x)$ defines a character $\varepsilon_x$ in $M(C(X))$, and the map $\varepsilon_X\colon X \to M(C(X))$, $x \mapsto \varepsilon_x$ is a homeomorphism. If $a \in A$, its *Gelfand transform* $\hat{a}\colon M(A) \to \mathbb{C}$, $\mu \mapsto \mu(a)$ is a continuous function on $M(A)$, and the map $\mathcal{G}\colon A \to C(M(A))$, $a \mapsto \hat{a}$ is a $*$-isomorphism of $C^*$-algebras, that preserves identities if $A$ is unital. These maps are *functorial* (or 'natural') in the sense that the following diagrams commute:

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\varepsilon_X \downarrow & & \downarrow \varepsilon_Y \\
M(C(X)) & \xrightarrow{\ MCf\ } & M(C(Y))
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\ \phi\ } & B \\
\mathcal{G}_A \downarrow & & \downarrow \mathcal{G}_B \\
C(M(A)) & \xrightarrow{\ CM\phi\ } & C(M(B))
\end{array}
$$

For instance, given a unital $*$-homomorphism $\phi\colon A \to B$, then for any $a \in A$ and $\nu \in M(B)$, we get

$$
((CM\phi \circ \mathcal{G}_A)a)\nu = ((CM\phi)\hat{a})\nu = \hat{a}((M\phi)\nu) = \hat{a}(\nu \circ \phi)
$$
$$
= \nu(\phi(a)) = \widehat{\phi(a)}(\nu) = ((\mathcal{G}_B \circ \phi)a)\nu,
$$

by unpacking the various transpositions.

This 'equivalence of categories' has several consequences. First of all, two commutative $C^*$-algebras are isomorphic if and only if their character spaces are homeomorphic. (If $\phi\colon A \to B$ and $\psi\colon B \to A$ are inverse $*$-isomorphisms, then $M\phi\colon M(B) \to M(A)$ and $M\psi\colon M(A) \to M(B)$ are inverse continuous proper maps.)

Secondly, the group of automorphisms $\mathrm{Aut}(A)$ of a commutative $C^*$-algebra $A$ is isomorphic to the group of homeomorphisms of its character space. Note that, since $A$ is commutative, there are no nontrivial inner automorphisms in $\mathrm{Aut}(A)$.

Thirdly, the topology of $X$ may be specified in terms of algebraic properties of $C_0(X)$. For instance, any *ideal* of $C_0(X)$ is of the form $C_0(U)$ where $U \subseteq X$ is an *open subset* (the closed set $X \setminus U$ being the zero set of this ideal).

If $Y \subseteq X$ is a *closed* subset of a compact space $X$, with inclusion map $j\colon Y \to X$, then $Cj\colon C(X) \to C(Y)$ is the restriction homomorphism (which is surjective, by Tietze's extension theorem). In general, $f\colon Y \to X$ is injective if and only if $Cf\colon C(X) \to C(Y)$ is surjective.

We may summarize several properties of the Gelfand–Naĭmark cofunctor with the following dictionary, adapted from [221, p. 24]:

| TOPOLOGY | ALGEBRA |
|---|---|
| locally compact space | $C^*$-algebra |
| compact space | unital $C^*$-algebra |
| compactification | unitization |
| continuous proper map | $*$-homomorphism |
| homeomorphism | automorphism |
| open subset | ideal |
| closed subset | quotient algebra |
| metrizable | separable |
| Baire measure | positive linear functional |

The $C^*$-algebra viewpoint also allows one to study the topology of non-Hausdorff spaces, such as arise in probing a continuum where points are unresolved: see the book by Landi on noncommutative spaces [138].

A commutative $C^*$-algebra has an abundant supply of characters, one for each point of the associated space. Looking ahead to noncommutative algebras, we can anticipate that characters will be fairly scarce, and we need not bother to search for points. There is, however, one role for points that survives in the noncommutative case: that of zero-dimensional elements of a homological skeleton or cell decomposition of a topological space. For that purpose, characters are not needed; we shall require functionals that are only *traces* on the algebra, but are not necessarily multiplicative.

## 1.2 The $\Gamma$ functor

Continuous functions determine a space's topology, but to do geometry we need at least a differentiable structure. Thus we shall assume from now on that our 'commutative space' is in fact a differential *manifold* $M$, of dimension $n$. For simplicity, we shall usually assume that $M$ is *compact*, even though this leaves aside important examples such as Minkowski space. (It turns out that noncommutative geometry has been developed so far almost entirely in the Euclidean signature, where compactness can be seen as a simplifying technical assumption. For the noncompact Euclidean case, see Chapter 9. How to adapt the theory to deal with spaces with indefinite metric is still an open problem, although there are by now several proposals available [133], [166], [203].)

The $C^*$-algebra $A = C(M)$ of continuous functions must then be replaced by the algebra $\mathcal{A} = C^\infty(M)$ of *smooth* functions on the manifold $M$. This is not a $C^*$-algebra, and although it is a Fréchet algebra in its natural locally convex topology, our tactic is to work with the dense subalgebra $\mathcal{A}$ of $A$ in a purely algebraic fashion. We think of $\mathcal{A}$ as the subspace of 'sufficiently regular' elements of $A$: see Section 3.4.

A character of $\mathcal{A}$ is a distribution $\mu$ on $M$ that is positive, since $\mu(a^*a) = |\mu(a)|^2 \geq 0$, and as such is a measure [95] that extends to a character of $C(M)$; hence $\mathcal{A}$ also determines the point-space $M$.

To study a given compact manifold $M$, one uses the category of (complex) *vector bundles* $E \xrightarrow{\pi} M$; its morphisms are bundle maps $\tau \colon E \to E'$ satisfying $\pi' \circ \tau = \pi$ and so defining fibrewise maps $\tau_x \colon E_x \to E'_x$ ($x \in M$) that are required to be linear. Given any vector bundle $E \xrightarrow{\pi} M$, write

$$\Gamma(E) := C^\infty(M, E)$$

for the space of *smooth sections* of $M$. If $\tau \colon E \to E'$ is a bundle map, the composition $\Gamma\tau \colon \Gamma(E) \to \Gamma(E')$, $s \mapsto \tau \circ s$ satisfies, for $a \in \mathcal{A}$, $x \in M$,

$$\Gamma\tau(sa)(x) = \tau_x(s(x)a(x)) = \tau_x(s(x))\,a(x) = (\Gamma\tau(s)a)(x)$$

so $\Gamma\tau(sa) = \Gamma\tau(s)a$; that is, $\Gamma\tau \colon \Gamma(E) \to \Gamma(E')$ is a morphism of (right) $\mathcal{A}$-modules. One may write either $as$ or $sa$ to denote the multiplication of a section $s$ by a function $a$, so $\Gamma(E)$ can also be regarded as a left $\mathcal{A}$-module, with appropriate changes of notation.

Vector bundles over $M$ admit operations such as duality, direct sum (i.e., Whitney sum) and tensor product; the $\Gamma$-functor carries these to analogous operations on $\mathcal{A}$-modules; for instance, if $E$, $E'$ are vector bundles over $M$, then

$$\Gamma(E \otimes E') \simeq \Gamma(E) \otimes_{\mathcal{A}} \Gamma(E'),$$

where the right hand side is formed by finite sums $\sum_j s_j \otimes s'_j$ subject to the relations $sa \otimes s' - s \otimes as' = 0$, for $a \in \mathcal{A}$. One can show that any $\mathcal{A}$-linear map from $\Gamma(E)$ to $\Gamma(E')$ is of the form $\Gamma\tau$ for a unique bundle map $\tau \colon E \to E'$.

It remains to identify the image of the $\Gamma$-functor. Note that if $E = M \times \mathbb{C}^r$ is a trivial bundle, then $\Gamma(E) = \mathcal{A}^r$ is a *free* $\mathcal{A}$-module. Since $M$ is compact, we can find nonnegative functions $\psi_1, \ldots, \psi_q \in \mathcal{A}$ with $\psi_1^2 + \cdots + \psi_q^2 = 1$ (a partition of unity) such that $E$ is trivial over the set $U_j$ where $\psi_j > 0$, for each $j$. If $f_{ij} \colon U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})$ are the transition functions for $E$, satisfying $f_{ik} f_{kj} = f_{ij}$ on $U_i \cap U_j \cap U_k$, then the functions $p_{ij} = \psi_i f_{ij} \psi_j$ (defined to be zero outside $U_i \cap U_j$) satisfy $\sum_k p_{ik} p_{kj} = p_{ij}$, and so assemble into a $qr \times qr$ matrix $p \in M_{qr}(\mathcal{A})$ such that $p^2 = p$. A section in $\Gamma(E)$, given locally by smooth functions $s_j \colon U_j \to \mathbb{C}^r$ such that $s_i = f_{ij} s_j$ on $U_i \cap U_j$, can be regarded as a column vector $s = (\psi_1 s_1, \ldots, \psi_q s_q)^t \in C^\infty(M)^{qr}$ satisfying $ps = s$. In this way, one identifies $\Gamma(E)$ with $p\mathcal{A}^{qr}$.

The *Serre–Swan theorem* [205] says that this is a two-way street: if $\mathcal{A} = C^\infty(M)$, then any right $\mathcal{A}$-module of the form $p\mathcal{A}^m$, for an idempotent $p \in M_m(\mathcal{A})$, is isomorphic to $\Gamma(E) = C^\infty(M, E)$ for some vector bundle $E$. The fibre at the point $\mu \in M = M(\mathcal{A})$ is the vector space $p\mathcal{A}^m \otimes_{\mathcal{A}} (\mathcal{A}/\ker \mu)$ whose (finite) dimension is the trace of the matrix $\mu(p) \in M_m(\mathbb{C})$.

In general, if $\mathcal{A}$ is any unital algebra, a right $\mathcal{A}$-module of the form $p\mathcal{A}^m$ is called a *finitely generated projective module*. We summarize by saying that $\Gamma$ is a (covariant) functor from the category of vector bundles over $M$ to the category of finitely generated projective modules over $C^\infty(M)$. The Serre–Swan theorem gives a recipe to construct an inverse functor going the other way, so that these categories are equivalent. (See the discussion by Brodzki [17], or Chapter 2 of [104], for more details in a modern style.)

What, then, is a *noncommutative vector bundle*? It is simply a finitely generated projective right module $\mathcal{E}$ for a (not necessarily commutative) algebra $\mathcal{A}$, which will generally be a dense subalgebra of a $C^*$-algebra $A$.

## 1.3 Hermitian metrics and spin$^c$ structures

Any complex vector bundle can be endowed (in many ways) with a Hermitian metric. The conventional practice is to define a positive definite sesquilinear form $(\cdot \mid \cdot)_x$ on each fibre $E_x$ of the bundle, which must 'vary smoothly with $x$'. The noncommutative point of view is to eliminate $x$, and what remains is a *pairing* $\mathcal{E} \times \mathcal{E} \to \mathcal{A}$ on a finitely generated projective right $\mathcal{A}$-module with values in the algebra $\mathcal{A}$ that is $\mathcal{A}$-linear in the second variable, hermitian and positive definite. In symbols:

$$\begin{aligned}
(r \mid s + t) &= (r \mid s) + (r \mid t), \\
(r \mid sa) &= (r \mid s)\, a, \\
(r \mid s) &= (s \mid r)^*, \\
(s \mid s) &> 0 \quad \text{for } s \neq 0,
\end{aligned} \tag{1.1}$$

for $r, s, t \in \mathcal{E}$, $a \in \mathcal{A}$. Notice the consequence $(rb \mid s) = b^* (r \mid s)$ if $b \in \mathcal{A}$.

With this structure, $\mathcal{E}$ is called a *pre-$C^*$-module* or 'prehilbert module'. More precisely, a pre-$C^*$-module over a dense subalgebra $\mathcal{A}$ of a $C^*$-algebra $A$ is a right $\mathcal{A}$-module $\mathcal{E}$ (not necessarily finitely generated or projective) with a pairing $\mathcal{E} \times \mathcal{E} \to \mathcal{A}$ satisfying (1.1). One can complete it in the norm

$$\|\!|s|\!\| := \sqrt{\|(s \mid s)\|}$$

where $\|\cdot\|$ is the $C^*$-norm of $A$; the resulting Banach space is then a *$C^*$-module*. In the case $\mathcal{E} = C^\infty(M, E)$, the completion is the Banach space of continuous sections $C(M, E)$. Indeed, in general this completion is *not* a Hilbert space. For instance, one can take $\mathcal{E} = \mathcal{A}$ itself, by defining $(a \mid b) := a^*b$; then $\|\!|a|\!\|$ equals the $C^*$-norm $\|a\|$, so the completion is the $C^*$-algebra $A$.

The free $\mathcal{A}$-module $\mathcal{A}^m$ is a pre-$C^*$-module in the obvious way, namely $(r \mid s) := \sum_{j=1}^m r_j^* s_j$. This column-vector scalar product also works for $p\mathcal{A}^m$ if $p = p^2 \in M_m(\mathcal{A})$, provided that $p = p^*$ also. If $q = q^2 \in M_m(\mathcal{A})$, one can always find a *projector* $p = p^2 = p^*$ in $M_m(\mathcal{A})$ that is similar and homotopic to $q$: see, for example, [104, Thm. 3.8]. (The choice of $p$ selects a particular Hermitian structure on the right module $q\mathcal{A}^m$.) Thus we shall always assume from now on that the idempotent $p$ is also selfadjoint.

One can similarly study *left* $\mathcal{A}$-modules. In fact, if $\mathcal{E}$ is any right $\mathcal{A}$-module, the *conjugate* space $\bar{\mathcal{E}}$ is a left $\mathcal{A}$-module: by writing $\bar{\mathcal{E}} = \{\bar{s} : s \in \mathcal{E}\}$, we can define $a\,\bar{s} := (sa^*)^-$. For $\mathcal{E} = p\mathcal{A}^m$, we get $\bar{\mathcal{E}} = \bar{\mathcal{A}}^m p$ where entries of $\bar{\mathcal{A}}^m$ are to be regarded as 'row vectors'.

**Morita equivalence.** Finitely generated projective $\mathcal{A}$-modules with $\mathcal{A}$-valued pairings play a rôle in noncommutative geometry as *mediating structures* that is partially hidden in commutative geometry: they allow the emergence of new algebras related, but not isomorphic, to $\mathcal{A}$. Consider the 'ket-bra' operators

$$|r\rangle\langle s| \colon \mathcal{E} \to \mathcal{E}, \quad t \mapsto r\,(s \mid t), \quad \text{for } r, s \in \mathcal{E}. \tag{1.2}$$

Composing two ket-bras yields a ket-bra:

$$|r\rangle\langle s| \cdot |t\rangle\langle u| = |r(s \mid t)\rangle\langle u| = |r\rangle\langle u\,(t \mid s)|,$$

so all finite sums of ket-bras form an algebra $\mathcal{B}$. Since $r\,(s \mid ta) = r\,(s \mid t)\,a$ for $a \in \mathcal{A}$, ket-bras act 'on the left' on $\mathcal{E}$ and commute with the right action of $\mathcal{A}$. If $\mathcal{E} = p\mathcal{A}^m$, then $\mathcal{B} = p\,M_m(\mathcal{A})\,p$. In this way, $\mathcal{E}$ becomes a '$\mathcal{B}$-$\mathcal{A}$-*bimodule*'.

If $\mathcal{A}$ is unital, one can regard $\mathcal{B}$ as $\mathcal{E} \otimes_{\mathcal{A}} \bar{\mathcal{E}}$, by $|r\rangle\langle s| \leftrightarrow r \otimes \bar{s}$. On the other hand, we can form $\bar{\mathcal{E}} \otimes_{\mathcal{B}} \mathcal{E}$, which is isomorphic to $\mathcal{A}$ as an $\mathcal{A}$-bimodule via $\bar{r} \otimes s \leftrightarrow (r \mid s)$. This is an instance of *Morita equivalence*. In general, we say that two unital algebras $\mathcal{A}$, $\mathcal{B}$ are *Morita-equivalent* if there is a $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{E}$ and an $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{F}$ such that

$$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \simeq \mathcal{B}, \quad \mathcal{F} \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathcal{A}, \tag{1.3}$$

as $\mathcal{B}$-$\mathcal{B}$- and $\mathcal{A}$-$\mathcal{A}$-bimodules respectively. With $\mathcal{E} = \mathcal{A}^m$ and $\mathcal{F} \simeq \bar{\mathcal{A}}^m$, we see that any full matrix algebra over $\mathcal{A}$ is Morita-equivalent to $\mathcal{A}$; nontrivial projectors over $\mathcal{A}$ offer a host of more 'twisted' examples of algebras that are equivalent to $\mathcal{A}$ in this sense.

The importance of Morita equivalence of two algebras is that their representations match. More precisely, suppose that there is a Morita equivalence of two algebras $\mathcal{A}$ and $\mathcal{B}$, implemented by a pair of bimodules $\mathcal{E}$, $\mathcal{F}$ as in (1.3). Then the functors $\mathcal{H} \mapsto \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ and $\mathcal{H}' \mapsto \mathcal{F} \otimes_{\mathcal{B}} \mathcal{H}'$ implement opposing correspondences between representation spaces of $\mathcal{A}$ and $\mathcal{B}$.

*Moral*: if we study an algebra $\mathcal{A}$ only through its representations, we must simultaneously study the various algebras Morita-equivalent to $\mathcal{A}$. In particular, we package together the commutative algebra $C^\infty(M)$ and the noncommutative algebra $M_n(C^\infty(M))$ for the purpose of doing geometry.

In the category of $C^*$-algebras (with or without unit element), one replaces finitely generated projective modules by arbitrary $C^*$-modules and obtains a much richer theory; see, for instance, [137], [181] and especially [175]. The notion analogous to (1.3) is called 'strong Morita equivalence'. In particular, let us note that two $C^*$-algebras $A$ and $B$ are strongly Morita equivalent whenever $A \otimes \mathcal{K} \simeq B \otimes \mathcal{K}$, where $\mathcal{K}$ is the elementary $C^*$-algebra of compact operators on a separable, infinite-dimensional Hilbert space [19].

**Spin$^c$ structures.** Returning once more to ordinary manifolds, suppose that $M$ is an $n$-dimensional orientable Riemannian manifold with a metric $g$ on its tangent bundle $TM$. We build a Clifford algebra bundle $\mathbb{C}\ell'(M) \to M$ whose fibres are full matrix algebras (over $\mathbb{C}$), as follows. If $n$ is even, $n = 2m$, then $\mathbb{C}\ell'_x(M) := \mathrm{Cl}(T_xM, g_x) \otimes_{\mathbb{R}} \mathbb{C} \simeq M_{2^m}(\mathbb{C})$ is the complexified Clifford algebra over the tangent space $T_xM$. If $n$ is odd, $n = 2m + 1$, the analogous fibre splits as $M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$, so we take only the *even* part of the complexified Clifford algebra: $\mathbb{C}\ell'_x(M) := \mathrm{Cl}^{\mathrm{even}}(T_xM) \otimes_{\mathbb{R}} \mathbb{C} \simeq M_{2^m}(\mathbb{C})$. The price we pay for this choice is that we lose the $\mathbb{Z}_2$-grading of the Clifford algebra bundle in the odd-dimensional case.

What we gain is that in all cases, the bundle $\mathbb{C}\ell'(M) \to M$ is a locally trivial field of (finite-dimensional) elementary $C^*$-algebras. Such a field is classified, up to equivalence, by a third-degree Čech cohomology class $\delta(\mathbb{C}\ell'(M)) \in H^3(M, \mathbb{Z})$ called the Dixmier–Douady class [67], [175]. Locally, one finds trivial bundles with fibres $S_x$ such that $\mathbb{C}\ell'_x(M) \simeq \mathrm{End}(S_x)$; the class $\delta(\mathbb{C}\ell'(M))$ is precisely the obstruction to patching them together (there is no obstruction to the existence of the algebra bundle $\mathbb{C}\ell'(M)$). It was shown by Plymen [171] that $\delta(\mathbb{C}\ell'(M)) = W_3(TM)$, the integral class that is the obstruction to the existence of a *spin$^c$ structure* in the conventional sense of a lifting of the structure group of $TM$ from $\mathrm{SO}(n)$ to $\mathrm{Spin}^c(n)$: see [149, Appendix D] for more information on $W_3(TM)$.

Thus $M$ admits spin$^c$ structures if and only if $\delta(\mathbb{C}\ell'(M)) = 0$. But in the Dixmier–Douady theory, $\delta(\mathbb{C}\ell'(M))$ is the obstruction to constructing a $B$-$A$-bimodule $\mathcal{S}$ that

implements a (strong) Morita equivalence between the $C^*$-algebras $A = C_0(M)$ and $B = C_0(M, \mathbb{C}\ell'(M))$. Let us paraphrase Plymen's redefinition of a spin$^c$ structure, in the spirit of noncommutative geometry.

**Definition 1.** Let $M$ be a Riemannian manifold, $A = C_0(M)$ and $B = C_0(M, \mathbb{C}\ell'(M))$. We say that the tangent bundle $TM$ *admits a spin$^c$ structure* if and only if it is orientable and $\delta(\mathbb{C}\ell'(M)) = 0$. In that case, a *spin$^c$ structure* on $TM$ is a pair $(\varepsilon, \mathcal{S})$ where $\varepsilon$ is an orientation on $TM$ and $\mathcal{S}$ is a $B$-$A$-equivalence bimodule.

Following an earlier terminology introduced by Atiyah, Bott and Shapiro [4] in their seminal paper on Clifford modules, the pair $(\varepsilon, \mathcal{S})$ is also called a *K-orientation* on $M$. Notice that $K$-orientability demands more than mere orientability in the cohomological sense. In any case, from now on we consider only orientable manifolds $M$ with a fixed orientation $\varepsilon$, so that $K$-orientability amounts to the existence of $\mathcal{S}$. We note in passing that Plymen's approach recovers earlier work of Karrer on Clifford actions [123], [191].

What is this equivalence bimodule $\mathcal{S}$? By the Serre–Swan theorem, it is of the form $\Gamma(S)$ for some complex vector bundle $S \to M$ that also carries an irreducible left action of the Clifford algebra bundle $\mathbb{C}\ell'(M)$. This is the *spinor bundle* whose existence displays the spin$^c$ structure in the conventional picture. We call $\Gamma(S) = C^\infty(M, S)$ the *spinor module*; it is an irreducible Clifford module in the terminology of [4], and has rank $2^m$ over $C^\infty(M)$ if $n = 2m$ or $2m + 1$.

Another matter is how to fit into this picture *spin structures* on $M$ (liftings of the structure group of $TM$ from $SO(n)$ to $Spin(n)$ rather than $Spin^c(n)$). These are distinguished by the availability of a *conjugation operator* $J$ on the spinors (which is antilinear); we shall take up this matter in Chapter 3.

*To summarize*: the language of bimodules and Morita equivalence gives us direct access to noncommutative (or commutative) vector bundles without invoking the concept of a 'principal bundle'. The concept of a noncommutative principal bundle is certainly available – see, for instance, [107], [110], [143] and especially [7] – but here we leave this matter aside.

## 1.4 The Dirac operator and the distance formula

As soon as a spinor module makes its appearance, one can introduce the *Dirac operator*. This is a selfadjoint first-order differential operator $\slashed{D}$ defined on the space $\mathcal{H} := L^2(M, S)$ of square-integrable spinors, whose domain includes the smooth spinors $\mathcal{S} = C^\infty(M, S)$. If $M$ is even-dimensional, there is a $\mathbb{Z}_2$-grading $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ arising from the grading of the Clifford algebra bundle $\Gamma(\mathbb{C}\ell(M))$, which in turn induces a grading of the Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$; let us call the grading operator $\Gamma$, so that $\Gamma^2 = 1$ and $\mathcal{H}^\pm$ are its $(\pm 1)$-eigenspaces. The Dirac operator is obtained by composing the natural *covariant derivative* on the modules $\mathcal{S}^\pm$ (or just on $\mathcal{S}$ in the

odd-dimensional case) with the *Clifford multiplication by* 1-*forms* that reverses the grading.

We repeat that in more detail. The Riemannian metric $g = [g_{ij}]$ defines isomorphisms $T_x M \simeq T_x^* M$ and induces a metric $g^{-1} = [g^{ij}]$ on the cotangent bundle $T^* M$. Via this isomorphism, we can redefine the Clifford algebra as the bundle with fibres $\mathbb{C}\ell_x'(M) := \mathrm{C}\ell(T_x^* M, g_x^{-1}) \otimes_{\mathbb{R}} \mathbb{C}$ (replacing $\mathrm{C}\ell$ by $\mathrm{C}\ell^{\mathrm{even}}$ when dim $M$ is odd). Let $\mathcal{A}^1(M) := \Gamma(T^* M)$ be the $\mathcal{A}$-module of 1-forms on $M$. The spinor module $\mathcal{S}$ is then a $\mathcal{B}$-$\mathcal{A}$-bimodule on which the algebra $\mathcal{B} = \Gamma(\mathbb{C}\ell'(M))$ acts irreducibly and obeys the anticommutation rule

$$\{\gamma(\alpha), \gamma(\beta)\} = 2g^{-1}(\alpha, \beta) = 2g^{ij}\alpha_i\beta_j \quad \text{for } \alpha, \beta \in \mathcal{A}^1(M). \qquad (1.4)$$

Here $\gamma \colon \mathcal{A}^1(M) \to \mathcal{B}$ denotes the action of $\mathcal{A}^1(M)$ on $\mathcal{H}$.

The metric $g^{-1}$ on $T^* M$ gives rise to a canonical *Levi-Civita connection* $\nabla^g \colon \mathcal{A}^1(M) \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{A}^1(M)$ that, as well as obeying the Leibniz rule

$$\nabla^g(\omega a) = (\nabla^g \omega)\, a + \omega \otimes da,$$

preserves the metric and is torsion-free. A 'spin$^c$ connection' is then a linear operator $\nabla^S \colon \Gamma(S) \to \Gamma(S) \otimes_{\mathcal{A}} \mathcal{A}^1(M)$ satisfying two Leibniz rules, one for the right action of $\mathcal{A}$ and the other, involving the Levi-Civita connection, for the left action of the Clifford algebra:

$$\begin{aligned} \nabla^S(\psi a) &= \nabla^S(\psi)\, a + \psi \otimes da, \\ \nabla^S(\gamma(\omega)\psi) &= \gamma(\nabla^g \omega)\, \psi + \gamma(\omega)\, \nabla^S \psi, \end{aligned} \qquad (1.5)$$

for $a \in \mathcal{A}$, $\omega \in \mathcal{A}^1(M)$, $\psi \in \mathcal{S}$. In the presence of a spin structure with conjugation operator $J$, we say $\nabla^S$ is the *spin connection* if it also commutes with $J$; this spin connection is unique [104, Sec. 9.3].

Once the spin connection is found, we define the Dirac operator as the composition $(-i)\gamma \circ \nabla^S$; more precisely, the local expression

$$\slashed{D} := -i\, \gamma(dx^j)\, \nabla^S_{\partial/\partial x^j} \qquad (1.6)$$

is independent of the coordinates and defines $\slashed{D}$ on the domain $\mathcal{S} \subset \mathcal{H}$. The factor $(-i)$ is needed for $\slashed{D}$ to be selfadjoint instead of skewadjoint, when we adopt the positive-definite (Euclidean) convention for the Clifford relations (1.4). One can check that this operator is symmetric; it extends to an unbounded selfadjoint operator on $\mathcal{H}$, also called $\slashed{D}$. Since $M$ is compact, the latter $\slashed{D}$ is a Fredholm operator and its kernel is finite-dimensional. On the orthogonal complement of $(\ker \slashed{D})$ we may define $\slashed{D}^{-1}$, which is a compact operator.

**The distance formula.** The Dirac operator may be characterized more simply by its Leibniz rule. Since the algebra $\mathcal{A}$ is represented on the spinor space $\mathcal{H}$ by multiplication

operators, we may form $\slashed{D}(a\psi)$, for $a \in \mathcal{A}$ and $\psi \in \mathcal{H}$. It is an easy consequence of (1.5) and (1.6) that

$$\slashed{D}(a\psi) = -i\,\gamma(da)\psi + a\,\slashed{D}\psi. \tag{1.7}$$

This is the rule that we need to keep in mind. We can equivalently write it as

$$[\slashed{D}, a] = -i\,\gamma(da).$$

In particular, since $a$ is smooth and $M$ is compact, the operator $\|[\slashed{D}, a]\|$ is *bounded*, and its norm is simply the sup-norm $\|da\|_\infty$ of the differential $da$. This also equals the *Lipschitz seminorm* of $a$, defined as

$$\|a\|_{\mathrm{Lip}} := \sup_{p \neq q} \frac{|a(p) - a(q)|}{d(p, q)},$$

where $d(p, q)$ is the geodesic distance between the points $p$ and $q$ of the Riemannian manifold $M$. This might seem to be an unwelcome return to the use of points in geometry; but in fact this simple observation (by Connes) led to one of the great coups of noncommutative geometry [36]. One can simply stand the previous formula on its head:

$$d(p, q) = \sup\{|a(p) - a(q)| : a \in C(M), \ \|a\|_{\mathrm{Lip}} \leq 1\},$$
$$= \sup\{|(\hat{p} - \hat{q})(a)| : a \in C(M), \ \|[\slashed{D}, a]\| \leq 1\}, \tag{1.8}$$

and one discovers that *the metric on the space of characters $M = M(\mathcal{A})$ is entirely determined by the Dirac operator*.

This is, of course, just a tautology in commutative geometry; but it opens the way forward, since it shows that what one must carry over to the noncommutative case is precisely this operator, or a suitable analogue. One still must deal with the scarcity of characters for noncommutative algebras. The lesson that (1.8) teaches [39] is that the *length element $ds$* is in some sense inversely proportional to $\slashed{D}$.

The ingredients for a reformulation of commutative geometry in algebraic terms are almost in place. We list them briefly: an algebra $\mathcal{A}$; a representation space $\mathcal{H}$ for $\mathcal{A}$; and a selfadjoint operator $\slashed{D}$ on $\mathcal{H}$. Additionally, a conjugation operator $J$, still to be discussed; and, in even-dimensional cases, a $\mathbb{Z}_2$-grading operator $\Gamma$ on $\mathcal{H}$. This package of four or five terms is called a *real spectral triple* or a *real K-cycle* or, more simply, a *spin geometry*. Our task will be to study, to exemplify, and where possible, to parametrize these geometries.

# Spectral triples on the Riemann sphere

We now undertake the construction of some spectral triples $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$ for a very familiar commutative manifold, the Riemann sphere $\mathbb{S}^2$. This is an even-dimensional Riemannian spin manifold, indeed it is the simplest nontrivial representative of that class. Nevertheless, the associated spectral triples are not completely transparent, and their construction is very instructive.

The sphere $\mathbb{S}^2$ can also be regarded as the complex projective line $\mathbb{C}P^1$, or as the compactified plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. As such, it is described by two charts, $U_N$ and $U_S$, that omit respectively the north and south poles, with the respective local complex coordinates

$$z = e^{-i\phi} \cot \tfrac{\theta}{2}, \quad \zeta = e^{i\phi} \tan \tfrac{\theta}{2},$$

related by $\zeta = 1/z$ on the overlap $U_N \cap U_S$. We write $q(z) := 1 + z\bar{z}$ for convenience. The Riemannian metric $g$ and the area form $\Omega$ are given by

$$g = d\theta^2 + \sin^2 \theta \, d\phi^2 = 4q(z)^{-2} \, dz \, d\bar{z} = 4q(\zeta)^{-2} \, d\zeta \, d\bar{\zeta},$$
$$\Omega = \sin \theta \, d\theta \wedge d\phi = 2i \, q(z)^{-2} \, dz \wedge d\bar{z} = 2i \, q(\zeta)^{-2} \, d\zeta \wedge d\bar{\zeta}.$$

## 2.1 Line bundles and the spinor bundle

Hermitian line bundles over $\mathbb{S}^2$ correspond to finitely generated projective modules over $\mathcal{A} := C^\infty(\mathbb{S}^2)$, of 'rank one'; these are of the form $\mathcal{E} = p\mathcal{A}^n$ where $p = p^2 = p^* \in M_n(\mathcal{A})$ is a projector of constant rank 1. (Equivalently, $\mathcal{E}$ is of rank one if $\mathrm{End}_\mathcal{A}(\mathcal{E}) \simeq \mathcal{A}$.) It turns out that it is enough to consider the case $p \in M_2(\mathcal{A})$. We follow the treatment of Mignaco *et al.* [160]; see also [104, Sec. 2.6].

Using Pauli matrices $\sigma_1, \sigma_2, \sigma_3$, we may write any projector in $M_2(\mathcal{A})$ as

$$p = \frac{1}{2} \begin{pmatrix} 1 + n_3 & n_1 - in_2 \\ n_1 + in_2 & 1 - n_3 \end{pmatrix} = \tfrac{1}{2}(1 + \vec{n} \cdot \vec{\sigma})$$

where $\vec{n}$ is a smooth function from $\mathbb{S}^2$ to $\mathbb{S}^2$. Any homotopy between two such functions yields a homotopy between the corresponding projectors $p$ and $q$; and one can then construct a unitary element $u \in M_4(\mathcal{A})$ such that $u(p \oplus 0)u^{-1} = q \oplus 0$. Thus inequivalent finitely generated projective modules are classified by the homotopy group $\pi_2(\mathbb{S}^2) = \mathbb{Z}$, the corresponding integer $m$ being the *degree* of the map $\vec{n}$. If $f(z) = (n_1 - in_2)/(1 - n_3)$ is the corresponding map on $\mathbb{C}_\infty$ after stereographic projection,

then $m$ is also the degree of $f$. As a representative degree-$m$ map, one could choose $f(z) = z^m$ or $f(z) = 1/z^m$.

Let us examine the projector corresponding to $f(z) = z$, of degree 1. It is

$$p_B := \frac{1}{1 + z\bar{z}} \begin{pmatrix} z\bar{z} & z \\ \bar{z} & 1 \end{pmatrix} = \frac{1}{1 + \zeta\bar{\zeta}} \begin{pmatrix} 1 & \bar{\zeta} \\ \zeta & \zeta\bar{\zeta} \end{pmatrix},$$

which is the well-known *Bott projector* that plays a key role in $K$-theory [13]. In general, if $m > 0$, suitable projectors for the modules $\mathcal{E}_{(m)}$, $\mathcal{E}_{(-m)}$ of degrees $\pm m$ are

$$p_m = \frac{1}{1 + (z\bar{z})^m} \begin{pmatrix} (z\bar{z})^m & z^m \\ \bar{z}^m & 1 \end{pmatrix}, \quad p_{-m} = \frac{1}{1 + (z\bar{z})^m} \begin{pmatrix} (z\bar{z})^m & \bar{z}^m \\ z^m & 1 \end{pmatrix}.$$

One can identify $\mathcal{E}_{(1)}$ with the space of sections of the tautological line bundle $L \to \mathbb{C}P^1$ (the fibre at the point $[v] \in \mathbb{C}P^1$ being the subspace $\mathbb{C}v$ of $\mathbb{C}^2$), and $\mathcal{E}_{(-1)}$ with the space of sections of its dual, the so-called hyperplane bundle $H \to \mathbb{C}P^1$. In general, the integer $(-m)$ is the *Chern class* of the corresponding line bundle [96].

Let us choose basic local sections $\sigma_{mN}(z)$, $\sigma_{mS}(\zeta)$ for the module $\mathcal{E}_{(m)}$. We take, for $m > 0$,

$$\sigma_{mN}(z) := \frac{1}{\sqrt{1 + (z\bar{z})^m}} \begin{pmatrix} z^m \\ 1 \end{pmatrix}, \quad \sigma_{mS}(\zeta) := \frac{1}{\sqrt{1 + (\zeta\bar{\zeta})^m}} \begin{pmatrix} 1 \\ \zeta^m \end{pmatrix},$$

normalized so that $(\sigma_{mN} \mid \sigma_{mN}) = (\sigma_{mS} \mid \sigma_{mS}) = 1$. A global section $\sigma = f_N \sigma_{mN} = f_S \sigma_{mS}$ is thus determined by a pair of functions $f_N(z)$ and $f_S(\zeta)$ that are related on the overlap $U_N \cap U_S$ by the gauge transformation

$$f_N(z) = (\bar{z}/z)^{m/2} f_S(z^{-1}). \tag{2.1}$$

**Definition 2.** The *spinor bundle* $S = S^+ \oplus S^-$ over $\mathbb{S}^2$ has rank two and is $\mathbb{Z}_2$-graded; the *spinor module* $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ over $\mathcal{A}$ is likewise graded by $\mathcal{S}^{\pm} := \Gamma(S^{\pm})$. With the chosen conventions, we have $\mathcal{S}^+ \simeq \mathcal{E}_{(1)}$, $\mathcal{S}^- \simeq \mathcal{E}_{(-1)}$. Thus a *spinor* can be regarded as a pair of functions on each chart, $\psi_N^{\pm}(z)$ and $\psi_S^{\pm}(\zeta)$, related by

$$\psi_N^+(z) = \sqrt{\bar{z}/z}\, \psi_S^+(z^{-1}), \quad \psi_N^-(z) = \sqrt{z/\bar{z}}\, \psi_S^-(z^{-1}). \tag{2.2}$$

**The spin connection.** This is the connection $\nabla^S$ on the spinor module $\mathcal{S}$ determined by the Leibniz rule

$$\nabla^S(\gamma(\alpha)\psi) = \gamma(\nabla^g \alpha)\, \psi + \gamma(\alpha)\, \nabla^S \psi, \tag{2.3}$$

where $\nabla^g$ is the *Levi-Civita connection* on the cotangent bundle, determined by

$$\begin{aligned}
\nabla^g_{q\partial_z}\left(\frac{dz}{q}\right) &= \bar{z}\,\frac{dz}{q}, & \nabla^g_{q\partial_z}\left(\frac{d\bar{z}}{q}\right) &= -\bar{z}\,\frac{d\bar{z}}{q}, \\
\nabla^g_{q\bar{\partial}_z}\left(\frac{dz}{q}\right) &= -z\,\frac{dz}{q}, & \nabla^g_{q\bar{\partial}_z}\left(\frac{d\bar{z}}{q}\right) &= z\,\frac{d\bar{z}}{q}.
\end{aligned} \tag{2.4}$$

Next, $\gamma(\alpha)\psi$ is the Clifford action of the 1-form $\alpha$ on the spinor $\psi$. The gamma-matrices are Pauli matrices:

$$\gamma^1 := \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 := \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

It is convenient to introduce the complex combinations $\gamma^\pm := \frac{1}{2}(\gamma^1 \pm i\gamma^2)$. The grading operator for the spinor module $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is then given by

$$\gamma^3 := -i\,\gamma^1\gamma^2 = \sigma^3 = [\gamma^+, \gamma^-] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and we note that $\gamma^\pm \gamma^3 = \mp\gamma^\pm$. The Clifford action of 1-forms must satisfy

$$\{\gamma(dz), \gamma(dz)\} = 2g^{-1}(dz, dz) = 0,$$
$$\{\gamma(dz), \gamma(d\bar{z})\} = 2g^{-1}(dz, d\bar{z}) = q(z)^2,$$

so we take simply $\gamma(dz) := q(z)\,\gamma^+$ and $\gamma(d\bar{z}) := q(z)\,\gamma^-$. (This choice eliminates the natural ambiguity of the matrix square root of $q(z)^2\,1_2$, and so is a gauge fixing.) The spin connection is then given by

$$\nabla^S_{q\partial_z} = q\partial_z + \tfrac{1}{2}\bar{z}\,\gamma^3, \quad \nabla^S_{q\bar{\partial}_z} = q\bar{\partial}_z - \tfrac{1}{2}z\,\gamma^3, \tag{2.5}$$

which is consistent with (2.3) and (2.4). These operators commute with $\gamma^3$, and thus act on the rank-one modules $\mathcal{S}^+$, $\mathcal{S}^-$ by

$$\nabla^\pm_{q\partial_z} = q\partial_z \pm \tfrac{1}{2}\bar{z}, \quad \nabla^\pm_{q\bar{\partial}_z} = q\bar{\partial}_z \mp \tfrac{1}{2}z. \tag{2.6}$$

## 2.2 The Dirac operator on the sphere $\mathbb{S}^2$

**Definition 3.** The *Dirac operator* $\slashed{D} := -i\,\gamma(dx^j)\nabla^S_{\partial_j}$ on $\mathbb{S}^2$ may be rewritten in complex coordinates as

$$\slashed{D} = -i\,\gamma(dz)\,\nabla^S_{\partial_z} - i\,\gamma(d\bar{z})\,\nabla^S_{\bar{\partial}_z} = -i\,\gamma(d\zeta)\,\nabla^S_{\partial_\zeta} - i\,\gamma(d\bar{\zeta})\,\nabla^S_{\bar{\partial}_\zeta}.$$

Recalling the form (2.5) of the spin connection, we get

$$\slashed{D} = -i\,\gamma^+\nabla^S_{q\partial_z} - i\,\gamma^-\nabla^S_{q\bar{\partial}_z} = -i\,\gamma^+(q\,\partial_z + \tfrac{1}{2}\bar{z}\,\gamma^3) - i\,\gamma^-(q\,\bar{\partial}_z - \tfrac{1}{2}z\,\gamma^3)$$
$$= -i\,(q\,\partial_z - \tfrac{1}{2}\bar{z})\,\gamma^+ - i\,(q\,\bar{\partial}_z - \tfrac{1}{2}z)\,\gamma^-.$$

**The $\eth$ operator.** At this point, it is handy to employ a first-order differential operator introduced by Newman and Penrose [164]:

$$\eth_z := (1 + z\bar{z})\, \partial_z - \tfrac{1}{2}\bar{z} = q\, \partial_z - \tfrac{1}{2}\bar{z} = q^{3/2} \cdot \partial_z \cdot q^{-1/2} \qquad (2.7)$$

and its complex conjugate $\bar{\eth}_z := q\, \bar{\partial}_z - \tfrac{1}{2}z$. Then

$$\slashed{D} = -i\,(\eth_z\, \gamma^+ + \bar{\eth}_z\, \gamma^-) = -i \begin{pmatrix} 0 & \eth_z \\ \bar{\eth}_z & 0 \end{pmatrix}. \qquad (2.8)$$

This operator is selfadjoint, since $\eth_z$ is skewadjoint:

$$\langle \phi^+ \mid \eth_z \psi^- \rangle = -\langle \bar{\eth}_z \phi^+ \mid \psi^- \rangle,$$

on the Hilbert space $L^2(\mathbb{C}, 2i\, q^{-2}\, dz \wedge d\bar{z})$, in view of $\eth_z = q^{3/2} \cdot \partial_z \cdot q^{-1/2}$. The scalar product of spinors is then given by

$$\langle \psi_1 \mid \psi_2 \rangle = \langle \psi_1^+ \mid \psi_2^+ \rangle + \langle \psi_1^- \mid \psi_2^- \rangle := \int_{\mathbb{C}} \left( \bar{\psi}_1^+ \psi_2^+ + \bar{\psi}_1^- \psi_2^- \right) \Omega.$$

$\slashed{D}$ thus extends to a selfadjoint operator on this Hilbert space of spinors, which we call $\mathcal{H} := L^2(\mathbb{S}^2, S)$. Moreover, $\gamma^3$ extends to a grading operator on $\mathcal{H}$ for which $\mathcal{H}^\pm = L^2(\mathbb{S}^2, S^\pm)$, and it is immediate that $\slashed{D}\gamma^3 = -\gamma^3\slashed{D}$.

**Definition 4.** The *conjugation operator $J$* on the Hilbert space $\mathcal{H}$ of spinors is defined as follows:

$$J \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} := \begin{pmatrix} -\bar{\psi}^- \\ \bar{\psi}^+ \end{pmatrix} \qquad (2.9)$$

To see that $J$ is well-defined, it suffices to recall that the gauge transformation rules for upper and lower half-spinors are conjugate (2.2). Clearly $J^2 = -1$ and $J$ is antilinear, indeed *antiunitary* in the sense that $\langle J\psi_1 \mid J\psi_2 \rangle = \langle \psi_2 \mid \psi_1 \rangle$ for all $\psi_1, \psi_2 \in \mathcal{H}$. Moreover, $J$ anticommutes with the grading: $J\gamma^3 = -\gamma^3 J$. Note that $J = -i\gamma^2 K$ where $K$ is complex conjugation, so $J$ is the charge conjugation operator on spinors.

Finally, $J$ *commutes* with the Dirac operator: $J\slashed{D} = \slashed{D}J$. Here it is convenient to introduce the antilinear adjoint operator $J^\dagger$, defined by $\langle \psi_1 \mid J^\dagger \psi_2 \rangle := \langle \psi_2 \mid J\psi_1 \rangle$; of course, $J^\dagger = J^{-1} = -J$ since $J$ is antiunitary. The desired identity $J\slashed{D}J^\dagger = \slashed{D}$ now follows from

$$J\slashed{D}J^\dagger \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = J\slashed{D} \begin{pmatrix} \bar{\psi}^- \\ -\bar{\psi}^+ \end{pmatrix} = i\, J \begin{pmatrix} -\eth_z \bar{\psi}^+ \\ \bar{\eth}_z \bar{\psi}^- \end{pmatrix} = -i \begin{pmatrix} \eth_z \psi^- \\ \bar{\eth}_z \psi^+ \end{pmatrix} = \slashed{D} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}.$$

In the Chapter 3 we shall see that the three signs that appear in the commutation relations for $J$, namely $J^2 = -1$, $J\gamma^3 = -\gamma^3 J$ and $J\slashed{D} = +\slashed{D}J$, are *characteristic of dimension two*.

**Definition 5.** We call the data set $(C^\infty(\mathbb{S}^2), L^2(\mathbb{S}^2, S), \slashed{D}; \gamma^3, J)$ the *fundamental spectral triple* for the commutative spin manifold $\mathbb{S}^2$.

**The Lichnerowicz formula.** This formula [12] relates the square of the Dirac operator on the spinor module to the spinor Laplacian; the difference between the two is one quarter of the scalar curvature $s$ of the underlying spin manifold: $\not{D}^2 = \Delta^S + \frac{1}{4}s$. For the sphere $\mathbb{S}^2$ with the metric already chosen, the scalar curvature (or Gaussian curvature) is $s = g^{ij} R^k_{ikj} = 2$, so that the Lichnerowicz formula in this case is just

$$\not{D}^2 = \Delta^S + \tfrac{1}{2}. \tag{2.10}$$

The spinor Laplacian is the generalized Laplacian [12] on the spinor module:

$$\Delta^S = -g^{ij} \left( \nabla^S_{\partial_i} \nabla^S_{\partial_j} - \Gamma^k_{ij} \nabla^S_{\partial_k} \right),$$

which in the isotropic basis $\{q\partial_z, q\bar{\partial}_z\}$ reduces to

$$\Delta^S = -q^2 \partial_z \bar{\partial}_z + \tfrac{1}{4} z\bar{z} + \tfrac{1}{2} q (z\,\partial_z - \bar{z}\,\bar{\partial}_z)\, \gamma^3.$$

On the other hand, from (2.7) one gets directly

$$\not{D}^2 = \begin{pmatrix} -\eth_z \bar{\eth}_z & 0 \\ 0 & -\bar{\eth}_z \eth_z \end{pmatrix} = \left( -q^2 \partial_z^2 \bar{\partial}_z^2 + \tfrac{1}{4} z\bar{z} + \tfrac{1}{2} \right) + \tfrac{1}{2} q (z\,\partial_z - \bar{z}\,\bar{\partial}_z)\, \gamma^3 = \Delta^S + \tfrac{1}{2}.$$

## 2.3 Spinor harmonics and the spectrum of $\not{D}$

The eigenspinors of $\not{D}$ can now be found by turning up appropriate matrix elements of well-known representations of SU(2); but a more pedestrian approach is quicker. These eigenspinors appear already in Newman and Penrose [164] under the name *spinor harmonics*, and were further studied by Goldberg *et al.* [97].

Their construction is based on two simple observations. The first is an elementary calculation with the $\eth$ operator:

$$\begin{aligned} \eth_z\big(q^{-l} z^r (-\bar{z})^s\big) &= (l + \tfrac{1}{2} - r) q^{-l} z^r (-\bar{z})^{s+1} + r q^{-l} z^{r-1} (-\bar{z})^s, \\ -\bar{\eth}_z\big(q^{-l} z^r (-\bar{z})^s\big) &= (l + \tfrac{1}{2} - s) q^{-l} z^{r+1} (-\bar{z})^s + s q^{-l} z^r (-\bar{z})^{s-1}, \end{aligned} \tag{2.11}$$

where $r, s \in \mathbb{N}$. The first is easily checked, the second follows by complex conjugation. One sees at once that suitable combinations of the functions $q^{-l} z^r (-\bar{z})^s$, with $l$ and $(r - s)$ held fixed, will form eigenvectors for the operator $\not{D}$ on account of its presentation (2.8).

Secondly, compatibility with gauge transformations of spinors (2.2) imposes restrictions on the exponents $l, r, s$. If $\phi(z) := \sum_{r,s \geq 0} a(r, s)\, q^{-l} z^r (-\bar{z})^s$, then

$$(\bar{z}/z)^{1/2} \phi(z^{-1}) = (-1)^{l + \frac{1}{2}} \sum_{r,s \geq 0} a(r, s)\, q^{-l} z^{l - \frac{1}{2} - r} (-\bar{z})^{l + \frac{1}{2} - s},$$

so that $\phi \in \mathscr{S}^+$ iff $l - \frac{1}{2} \in \mathbb{N}$, and $a(r, s) \neq 0$ only for $r = 0, 1, \ldots, l - \frac{1}{2}$ and $s = 0, 1, \ldots, l + \frac{1}{2}$. To have $\phi \in \mathscr{S}^-$, interchange the restrictions on $r$ and $s$.

If $m := r - s \pm \frac{1}{2}$, the corresponding restriction is $m = -l, -l+1, \ldots, l-1, l$, a very familiar sight in the theory of angular momentum; but with the important difference that here $l$ and $m$ are *half-integers but not integers*, so the corresponding matrix elements do not drop to matrix elements of representations of SO(3).

We can now display the spinor harmonics. They form two families, $Y_{lm}^+$ and $Y_{lm}^-$, corresponding to upper and lower spinor components; they are indexed by

$$l \in \mathbb{N} + \tfrac{1}{2} = \left\{ \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots \right\}, \quad m \in \{-l, -l+1, \ldots, l-1, l\},$$

and the formulas are

$$Y_{lm}^+(z) := C_{lm}\, q^{-l} \sum_{r-s=m-\frac{1}{2}} \binom{l-\frac{1}{2}}{r}\binom{l+\frac{1}{2}}{s} z^r (-\bar{z})^s,$$

$$Y_{lm}^-(z) := C_{lm}\, q^{-l} \sum_{r-s=m+\frac{1}{2}} \binom{l+\frac{1}{2}}{r}\binom{l-\frac{1}{2}}{s} z^r (-\bar{z})^s, \tag{2.12}$$

where the normalization constants $C_{lm}$ are defined as

$$C_{lm} := (-1)^{l-m} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)!\,(l-m)!}{(l+\frac{1}{2})!\,(l-\frac{1}{2})!}}.$$

**Eigenspinors.** The coefficients in (2.12) are chosen so as to satisfy

$$\eth_z Y_{lm}^- = \left(l + \tfrac{1}{2}\right) Y_{lm}^+ \quad \text{and} \quad \bar{\eth}_z Y_{lm}^+ = -\left(l + \tfrac{1}{2}\right) Y_{lm}^-.$$

in view of (2.11). If we then define normalized spinors by

$$Y_{lm}' := \frac{1}{\sqrt{2}} \begin{pmatrix} Y_{lm}^+ \\ i\,Y_{lm}^- \end{pmatrix}, \quad Y_{lm}'' := \frac{1}{\sqrt{2}} \begin{pmatrix} -Y_{lm}^+ \\ i\,Y_{lm}^- \end{pmatrix},$$

we get an orthonormal family of eigenspinors for the Dirac operator:

$$\slashed{D} Y_{lm}' = \left(l + \tfrac{1}{2}\right) Y_{lm}', \quad \slashed{D} Y_{lm}'' = -\left(l + \tfrac{1}{2}\right) Y_{lm}'',$$

where the eigenvalues are nonzero integers and each eigenvalue $\pm\left(l + \frac{1}{2}\right)$ has multiplicity $(2l + 1)$. In fact, these are *all* the eigenvalues of $\slashed{D}$; for that, we need the following completeness result, established in [97]:

$$\sum_{l,m} \bar{Y}_{lm}^\pm(z)\, Y_{lm}^\pm(z') = \delta(\phi - \phi')\,\delta(\cos\theta - \cos\theta').$$

Consequently, the spinors $\left\{ Y_{lm}', Y_{lm}'' : l \in \mathbb{N} + \frac{1}{2}, m \in \{-l, \ldots, l\} \right\}$ form an orthonormal basis for the Hilbert space $\mathscr{H} = L^2(\mathbb{S}^2, S)$.

**The spectrum.**  We have thus computed the *spectrum* of the Dirac operator:

$$\mathrm{sp}(\slashed{D}) = \big\{ \pm \big(l + \tfrac{1}{2}\big) : l \in \mathbb{N} + \tfrac{1}{2} \big\} = \mathbb{Z} \setminus \{0\}, \tag{2.13}$$

with the aforementioned multiplicities $(2l + 1)$. Notice that, since the zero eigenvalue is missing, the Dirac operator $\slashed{D}$ is invertible and *it has index zero*.

   The Lichnerowicz formula (2.10) gives the spectrum of the spinor Laplacian,

$$\mathrm{sp}(\Delta^S) = \big\{ l^2 + l - \tfrac{1}{4} : l \in \mathbb{N} + \tfrac{1}{2} \big\},$$

with respective multiplicities $2(2l + 1)$. There is also a Casimir operator $C$ with the same eigenspaces and respective eigenvalues $l^2 + l$, so that $\slashed{D}^2 = C + \tfrac{1}{4} = C + \tfrac{1}{8}s$; this variant of the Lichnerowicz formula appears in the theory of Dirac operators on symmetric spaces [89, Sec. 3.5].

## 2.4 Twisted spinor modules

To define other spectral triples over $\mathcal{A} = C^\infty(M)$, we may *twist* the spinor module $\mathcal{S}$ by tensoring it with some other finitely generated projective $\mathcal{A}$-module $\mathcal{E}$, the Clifford action on $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}$ being given by

$$\gamma(\alpha)(\psi \otimes \sigma) := (\gamma(\alpha)\psi) \otimes \sigma \quad \text{for } \psi \in \mathcal{S}, \ \sigma \in \mathcal{E}.$$

We call $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}$, with this action of the algebra $\Gamma(\mathbb{C}\ell'(M))$, a *twisted spinor module*.

   We show, with our example $\mathcal{A} = C^\infty(\mathbb{S}^2)$, how one can create new spectral triples by twisting the fundamental one. However, these triples will not always respect the 'real structure' $J$, as we shall see.

   We examine first the case where $\mathcal{E} = \mathcal{E}_{(m)}$ is a module of sections of a complex line bundle of first Chern class $(-m)$. Then the twisted spinor module is also $\mathbb{Z}_2$-graded; indeed, $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}_{(m)} \simeq \mathcal{E}_{(m+1)} \oplus \mathcal{E}_{(m-1)}$.

**The twisted Dirac operator.**  The half-spinor modules $\mathcal{S}^{\pm} = \mathcal{E}_{(\pm 1)}$ have connections $\nabla^{\pm}$ given by (2.6). Now $\mathcal{E}_{(m)} \simeq \mathcal{E}_{(1)} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{E}_{(1)}$ ($m$ times) if $m > 0$ and $\mathcal{E}_{(m)} \simeq \mathcal{E}_{(-1)} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{E}_{(-1)}$ ($|m|$ times) if $m < 0$, so we can define a connection $\nabla^{(m)}$ on $\mathcal{E}_{(m)}$ by

$$\nabla^{(m)}(s_1 \otimes \cdots \otimes s_{|m|}) := \sum_{j=1}^{|m|} s_1 \otimes \cdots \otimes \nabla^{\pm}(s_j) \otimes \cdots \otimes s_{|m|},$$

and from (2.6) and the Leibniz rule it follows that

$$\nabla^{(m)}_{q\partial_z} = q\partial_z + \tfrac{1}{2}m\bar{z}, \quad \nabla^{(m)}_{q\bar{\partial}_z} = q\bar{\partial}_z - \tfrac{1}{2}mz.$$

On $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}_{(m)}$, we define the *twisted spin connection* $\widetilde{\nabla}^S := \nabla^S \otimes 1 + 1 \otimes \nabla^{(m)}$. We obtain

$$\widetilde{\nabla}^S_{q\partial_z} = q\partial_z + \tfrac{1}{2}\bar{z}(m + \gamma^3), \quad \widetilde{\nabla}^S_{q\bar{\partial}_z} = q\bar{\partial}_z - \tfrac{1}{2}z(m + \gamma^3).$$

The corresponding Dirac operator is

$$\begin{aligned}
\slashed{D}_m &= -i\gamma^+ \widetilde{\nabla}^S_{q\partial_z} - i\gamma^- \widetilde{\nabla}^S_{q\bar{\partial}_z} \\
&= -i\left(q\partial_z + \tfrac{1}{2}(m-1)\bar{z}\right)\gamma^+ - i\left(q\bar{\partial}_z - \tfrac{1}{2}(m+1)z\right)\gamma^- \\
&= -i(\eth_z + \tfrac{1}{2}m\bar{z})\gamma^+ - i(\bar{\eth}_z - \tfrac{1}{2}mz)\gamma^-,
\end{aligned}$$

or more pictorially,

$$\slashed{D}_m = \begin{pmatrix} 0 & \slashed{D}_m^- \\ \slashed{D}_m^+ & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & \eth_z + \tfrac{1}{2}m\bar{z} \\ \bar{\eth}_z - \tfrac{1}{2}mz & 0 \end{pmatrix}.$$

This extends to a selfadjoint operator on the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}_{(m)}$ where $\mathcal{H}_{(m)}^{\pm} = L^2(\mathbb{S}^2, L^{m\pm 1})$.

**Computation of the index.** Notice that

$$\slashed{D}_m^+ = -i\, q^{(m+3)/2} \cdot \bar{\eth}_z \cdot q^{-(m+1)/2},$$

so that a half-spinor $\psi^+$ lies in $\ker \slashed{D}_m^+$ if and only if $\psi_N^+(z) = q(z)^{(m+1)/2} a(z)$ where $a$ is an entire holomorphic function. The gauge transformation rule (2.1) shows that the function $\psi_S^+(z^{-1}) = (z/\bar{z})^{(m+1)/2}\psi_N^+(z)$ is regular at $z = \infty$ only if either $a = 0$, or $m < 0$ and $a(z)$ is a polynomial of degree $< |m|$. Thus $\dim \ker \slashed{D}_m^+ = |m|$ if $m < 0$ and equals 0 for $m \geq 0$. A similar argument shows that $\dim \ker \slashed{D}_m^- = m$ if $m > 0$ and is 0 for $m \leq 0$. We conclude that $\slashed{D}_m^+$ is a Fredholm operator on $\mathcal{H}_{(m)}$, whose index is

$$\text{index}\,\slashed{D}_m^+ := \dim \ker \slashed{D}_m^+ - \dim \ker \slashed{D}_m^- = -m,$$

which equals the first Chern class of the twisting bundle.

**Incompatibility with the real structure.** The twisting by $\mathcal{E}_{(m)}$ loses the property of commutation with the charge conjugation $J$ (2.9). In fact, it is easy to check that

$$J\slashed{D}_m J^\dagger = -i\begin{pmatrix} 0 & \eth_z - \tfrac{1}{2}m\bar{z} \\ \bar{\eth}_z + \tfrac{1}{2}mz & 0 \end{pmatrix} = \slashed{D}_{-m}.$$

In conventional language, we could say that the twisted spinor bundle $S \otimes L^m$ is associated to a *spin$^c$ structure* on $T\mathbb{S}^2$, and that this is a *spin structure* only if $m = 0$. Conjugation by $J$ exchanges the spin$^c$ structures, fixing only the spin structure; this exemplifies the general fact [40] that commutation (or anticommutation) of $\slashed{D}$ with $J$

picks out a spin structure when these are available. (The charge conjugation operator is tightly constrained by its interaction with the coordinate algebra $\mathcal{A}$ and the Clifford algebra $\mathcal{B}$, whatever spinor module is: see [104, Thm. 9.6]; for the two-sphere, $J$ is unique up to multiplication by a trivial phase factor.) In view of these circumstances, we shall say that $J$ defines a *real structure* on $(\mathcal{A}, \mathcal{H})$.

In summary, $(C^\infty(\mathbb{S}^2), \mathcal{H}_{(m)}, \displaystyle{\not}D_m; \gamma^3)$ is a (complex) spectral triple, but is not a 'real spectral triple' if $m \neq 0$.

## 2.5 A reducible spectral triple

The twisted spinor modules discussed above are irreducible for the action of the Clifford algebra $\mathcal{B} = \Gamma(\mathbb{C}\ell'(\mathbb{S}^2))$. On the other hand, $\mathcal{B}$ acts reducibly on the algebra of differential forms $\mathcal{A}^\bullet(\mathbb{S}^2)$ by

$$\gamma(\alpha)\,\omega := \alpha \wedge \omega + \iota(\alpha^\sharp)\omega \quad \text{for } \alpha \in \mathcal{A}^1(\mathbb{S}^2),$$

where $\alpha^\sharp$ is the vector field determined by $\alpha^\sharp(f) := g^{-1}(\alpha, df)$, $f \in \mathcal{A}$. On the algebra of forms we can use the *Hodge star operator*, defined as the involutive $\mathcal{A}$-module isomorphism determined by $\star 1 = -i\,\Omega$, $\star d\theta = i \sin\theta\, d\phi$ (the phase factors are inserted to make $\star\star = 1$); in complex coordinates,

$$\star 1 = 2q^{-2}\, dz \wedge d\bar{z}, \quad \star(q^{-1}\, dz) = i\, q^{-1}\, d\bar{z}.$$

The *codifferential* $\delta = -\star d\star$ is the adjoint of the differential $d$ with respect to the scalar product of forms:

$$\langle \alpha \mid \beta \rangle = i(-1)^{k(k+1)/2} \int_{\mathbb{S}^2} \bar{\alpha} \wedge \star\beta \quad \text{for } \alpha, \beta \in \mathcal{A}^k(\mathbb{S}^2), \tag{2.14}$$

with which $\mathcal{A}^\bullet(\mathbb{S}^2) = \mathcal{A}^0(\mathbb{S}^2) \oplus \mathcal{A}^1(\mathbb{S}^2) \oplus \mathcal{A}^2(\mathbb{S}^2)$ may be completed to a Hilbert space $\mathcal{H} := L^{2,0}(\mathbb{S}^2) \oplus L^{2,1}(\mathbb{S}^2) \oplus L^{2,2}(\mathbb{S}^2)$.

**The Hodge–Dirac operator.** One can construct a Dirac operator on this Hilbert space by twisting, along the following lines. One can identify $\mathcal{A}^\bullet(\mathbb{S}^2)$ with $\mathcal{S} \otimes_\mathcal{A} \mathcal{S}'$ as $\mathcal{B}$-$\mathcal{A}$-s, where $\mathcal{S}'$ denotes the spinor module with the opposite grading: $(\mathcal{S}')^\pm = \mathcal{S}^\mp$. A detailed comparison of these bimodules and their Dirac operators is given in [209]. The spin connection on $\mathcal{S}'$ is given – compare (2.5) – by

$$\nabla^{S'}_{q\partial_z} = q\partial_z - \tfrac{1}{2}\bar{z}\,\gamma^3, \quad \nabla^{S'}_{q\bar{\partial}_z} = q\bar{\partial}_z + \tfrac{1}{2}z\,\gamma^3,$$

and $\widetilde{\nabla} := \nabla^S \otimes 1 + 1 \otimes \nabla^{S'}$ gives the tensor product connection on $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{S}'$. The Dirac operator on this twisted module is then

$$\widetilde{\slashed{D}} := -i(\gamma^+ \otimes 1)\, \widetilde{\nabla}_{q\partial_z} - i(\gamma^- \otimes 1)\, \widetilde{\nabla}_{q\bar{\partial}_z}$$
$$= \slashed{D} \otimes 1 + \gamma^+ \otimes \nabla^{S'}_{q\partial_z} + \gamma^- \otimes \nabla^{S'}_{q\bar{\partial}_z}.$$

The Lichnerowicz formula for this operator is [209]:

$$\widetilde{\slashed{D}}^2 = \widetilde{\Delta} + \tfrac{1}{2} - \tfrac{1}{2}(\gamma^3 \otimes \gamma^3), \tag{2.15}$$

where the term $-\tfrac{1}{2}(\gamma^3 \otimes \gamma^3)$ is the 'twisting curvature' [12].

Ugalde [209] has shown that, via an appropriate $\mathcal{A}$-module isomorphism $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{S}' \simeq \mathcal{A}^{\bullet}(\mathbb{S}^2)$, the corresponding operator on $\mathcal{A}^{\bullet}(\mathbb{S}^2)$ is precisely the operator $-i(d - \delta)$, that we call the 'Hodge–Dirac operator'. Its square is the *Hodge Laplacian* $\Delta^H :=$ $-(d - \delta)^2 = d\delta + \delta d$. Under the aforementioned isomorphism, (2.15) transforms to $-(d - \delta)^2 = \Delta^H + \tfrac{1}{2} - \tfrac{1}{2}$. Another presentation of the Hodge–Dirac operator is given in [104, Sec. 9.B].

The eigenforms for the Hodge Laplacian on spheres have been determined by Folland [87]. For $n = 2$, the eigenvalues of $\Delta^H$ are $\{l(l+1) : l \in \mathbb{N}\}$ with multiplicities $4(2l + 1)$ for $l = 1, 2, 3, \ldots$; for $l = 0$, there is a 2-dimensional kernel of harmonic forms, generated by 1 and $i\,\Omega$. The other eigenforms are interchanged by $d$ and $\delta$, and so may be combined to get a complete set of eigenvectors for $-i(d - \delta)$; this yields

$$\mathrm{sp}(-i(d - \delta)) = \left\{ \pm \sqrt{l(l + 1)} : l \in \mathbb{N} \right\}, \tag{2.16}$$

with respective multiplicities $2(2l + 1)$.

**Grading and real structure.** There are two $\mathbb{Z}_2$-grading operators at our disposal on the Hilbert space $\mathcal{H}$ of forms: the even/odd form-degree grading $\varepsilon$ and the Hodge star operator $\star$. In differential-geometric language, these correspond to selecting the *de Rham complex* or the *signature complex* as the object of interest [96]. The Dirac operator $-i(d - \delta)$ is odd for either grading.

To distinguish these $\mathbb{Z}_2$-gradings on the spectral triple $(\mathcal{A}, \mathcal{H}, -i(d - \delta))$, we look for a real structure $J$: an antilinear isometry, satisfying $J^2 = -1$ and $JaJ^{\dagger} = \bar{a}$ if $a \in C^{\infty}(\mathbb{S}^2)$, that commutes with $-i(d - \delta)$ and anticommutes with the grading operator. In particular, $J$ must preserve the two-dimensional kernel. Since the harmonic forms have even degree, any such $J$ cannot anticommute with the grading $\varepsilon$. However, it turns out that the eigenspaces for $-i(d - \delta)$ can be split into selfdual and antiselfdual subspaces of dimension $2l + 1$ each. One can then find a conjugation $J$ that does anticommute with the Hodge star operator: $J \star J^{\dagger} = -\star$. This yields a real spectral triple:

$$(C^{\infty}(\mathbb{S}^2), L^{2,\bullet}(\mathbb{S}^2), -i(d - \delta); \star, J).$$

This is the 'Dirac–Kähler' geometry studied by Mignaco *et al.* [160]. It also appears prominently in [90].

# 3

# Real spectral striples: the axiomatic foundation

Having exemplified how differential geometry may be made algebraic in the commutative case of Riemannian spin manifolds, we now extract the essential features of this formulation, with a view to relaxing the constraint of commutativity on the underlying algebra. We shall follow quite closely the treatment of Connes in [40], [41], wherein an axiomatic scheme for noncommutative geometries is set forth. Indeed, one could say that these lectures are essentially an extended meditation on those 'axioms'.

## 3.1 The data set

The fundamental object of study is a *spectral triple*, also known as an (unbounded) *K-cycle* because it is a building block for *K*-homology: see [113]. This consists of three pieces of data $(\mathcal{A}, \mathcal{H}, D)$, sometimes accompanied by other data $\Gamma$ and $J$, satisfying several compatibility conditions which we formulate as axioms. If $\Gamma$ is present, we say that the *K*-cycle is *even*, otherwise it is *odd*. If $J$ is present, we say the *K*-cycle is *real*; if not, we can call it 'complex'. We shall, however, concentrate on the real case.

**Definition 6.** An *even, real, spectral triple* is of a set of five objects $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$ of the following types:

1. $\mathcal{A}$ is a unital, dense $*$-subalgebra of a $C^*$-algebra $A$;

2. $\mathcal{H}$ is a *Hilbert space* carrying a faithful representation $\pi$ of $A$ by bounded operators;

3. $D$ is a *selfadjoint operator* on $\mathcal{H}$, with compact resolvent, such that for each $a \in \mathcal{A}$, the commutator $[D, \pi(a)]$ is a bounded operator on $\mathcal{H}$;

4. $\Gamma$ is a *selfadjoint unitary operator* on $\mathcal{H}$, so that $\Gamma^2 = 1$;

5. $J$ is an *antilinear isometry* of $\mathcal{H}$ onto itself.

Before introducing the further relations and properties that these objects must satisfy, we make some comments on the data themselves.

1. In principle, no topology on $\mathcal{A}$ need be given. However, it often happens that $\mathcal{A}$ is a locally convex topological vector space whose topology is finer than that of the $C^*$-norm of $A$; in particular, in many examples $\mathcal{A}$ is a Fréchet space. We

may then ask that $\mathcal{A}$ be a *pre-C*-algebra*, namely, that the inclusion $\mathcal{A} \hookrightarrow A$ be continuous and that $\mathcal{A}$ be *stable under the holomorphic functional calculus* of the $C^*$-algebra $A$. This means that for any $a \in \mathcal{A}$ and any function $f$ holomorphic in a neighbourhood of the spectrum $\mathrm{sp}(a)$ in $A$, the element $f(a) \in A$ actually belongs to $\mathcal{A}$. Here $f(a)$ is the Dunford integral, where $\gamma$ winds once positively around $\mathrm{sp}(a)$:

$$f(a) := \frac{1}{2\pi i} \oint_\gamma f(\zeta)(\zeta - a)^{-1}\, d\zeta.$$

A Fréchet algebra $\mathcal{A}$ is a pre-$C^*$-algebra if and only if whenever $a \in \mathcal{A}$ is invertible in $A$, the inverse $a^{-1}$ lies in $\mathcal{A}$: see [104, Sec. 3.8].

This happens whenever $\mathcal{A} = \bigcap_{k=1}^\infty \mathrm{Dom}(L^k)$, where $L$ is a densely defined derivation of $A$, since $L(a^{-1}) = -a^{-1} L(a)\, a^{-1}$ for any derivation; in this case, $\mathcal{A}$ is the set of smooth elements of $A$ under the action of the one-parameter group of automorphisms generated by $L$. Or $\mathcal{A}$ could be the common smooth domain of a Lie algebra of such derivations. In the case $\mathcal{A} = C^\infty(M)$, a smooth nonvanishing function has a smooth reciprocal.

When $\mathcal{A}$ is a Fréchet pre-$C^*$-algebra, the $K$-theories of $\mathcal{A}$ and of $A$ are the same. That is to say, the inclusion $j \colon \mathcal{A} \to A$ induces isomorphisms $j_* \colon K_0(\mathcal{A}) \to K_0(A)$ and $j_* \colon K_1(\mathcal{A}) \to K_1(A)$. This was proved by Bost [15], who first showed that Fréchet pre-$C^*$-algebras admit a useful topological-$K_1$ functor.

2. When $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$, we shall usually write $a\xi := \pi(a)\xi$. On a few occasions, though, we shall need to refer explicitly to the representation $\pi$.

3. That $D$ has 'compact resolvent' means that $(D - \lambda)^{-1}$ is compact, whenever $\lambda \notin \mathbb{R}$. Equivalently, $D$ has a finite-dimensional kernel, and the inverse $D^{-1}$ (defined on the orthogonal complement of this kernel) is compact. In particular, $D$ has a discrete spectrum of eigenvalues of finite multiplicity. This generalizes the case of a Dirac operator on a *compact* spin manifold; thus the spectral triples discussed here represent 'noncommutative compact manifolds'. Noncompact manifolds can be treated in a parallel manner by deleting the requirement that the algebra $\mathcal{A}$ have a unit, whereupon we require only that for each $a \in \mathcal{A}$, the operators $a(D - \lambda)^{-1}$ be compact [39]. Several examples of such *nonunital* spectral triples, modelling 'noncompact noncommutative spin geometries' are now available [91], [93], [101], [177], [178]: see Chapter 9.

On the basis of the distance formula (1.8), we shall interpret the inverse of $D$ as a *length element*. Since $D$ is generally not a positive operator, we use the inverse of its modulus $|D| = (D^2)^{1/2}$; we shall write $ds := |D|^{-1}$. (Actually, the point of view advocated in [40] is that one should treat $ds$ as an abstract symbol adjoined to the algebra $\mathcal{A}$ and consider $|D|^{-1}$ as its representative on $\mathcal{H}$; but we shall ignore this distinction here.)

4. The $\mathbb{Z}_2$-*grading operator* $\Gamma$, available for even spectral triples, splits the Hilbert space as $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^\pm$ is the $(\pm 1)$-eigenspace of $\Gamma$. In this case, we suppose that the representation of $\mathcal{A}$ on $\mathcal{H}$ is *even* and that the operator $D$ is *odd*, that is, $a\Gamma = \Gamma a$ for $a \in \mathcal{A}$ and $D\Gamma = -\Gamma D$. We display this symbolically as

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix},$$

where $D^+ \colon \mathcal{H}^+ \to \mathcal{H}^-$ and $D^- \colon \mathcal{H}^- \to \mathcal{H}^+$ are adjoint to each other.

5. The *real structure* $J$ must satisfy $J^2 = \pm 1$ and commutation relations $JD = \pm DJ$, $J\Gamma = \pm \Gamma J$; for the signs, see Axiom 7 below. Its adjoint is $J^\dagger = J^{-1} = \pm J$. The recipe

$$\pi^\circ(b) := J\pi(b^*)J^\dagger \tag{3.1}$$

defines an antirepresentation of $\mathcal{A}$, that is, it reverses the product. It is convenient to think of $\pi^\circ$ as a true representation of the *opposite algebra* $\mathcal{A}^\circ$, consisting of elements $\{a^\circ : a \in \mathcal{A}\}$ with product $a^\circ b^\circ = (ba)^\circ$. Thus we shall usually abbreviate (3.1) to $b^\circ = Jb^*J^\dagger$. The important property that we require is that *the representations $\pi$ and $\pi^\circ$ commute*; that is,

$$[a, b^\circ] = [a, Jb^*J^\dagger] = 0 \quad \text{for all } a, b \in \mathcal{A}. \tag{3.2}$$

When $\mathcal{A}$ is commutative, we may demand also that $J\pi(b^*)J^\dagger = \pi(b)$, making (3.2) automatic. In this case, our data yield a 'commutative spectral triple' [40]. This requires that $\mathcal{A}$ act as scalar multiplication operators on the spinor space, as exemplified in Chapter 2.

The stage is set. We now deal with the further conditions needed to ensure that these data underlie a *spin geometry*.

## 3.2 Infinitesimals and dimension

**Axiom 1** (Dimension). *There is an integer n, the* metric dimension *of the spectral triple, such that the length element $ds := |D|^{-1}$ is an infinitesimal of order $1/n$.*

By 'infinitesimal' we mean simply a *compact operator* on $\mathcal{H}$. Since the days of Leibniz, an infinitesimal is conceptually a nonzero quantity smaller than any positive $\varepsilon$. Since we work on the arena of an infinite-dimensional Hilbert space, we may forgive the violation of the requirement $T < \varepsilon$ over a finite-dimensional subspace (that may depend on $\varepsilon$). $T$ must then be an operator with discrete spectrum, with any nonzero $\lambda$ in $\text{sp}(T)$ having finite multiplicity; in other words, the operator $T$ must be compact.

The *singular values* of $T$, i.e., the eigenvalues of the positive compact operator $|T| := (T^*T)^{1/2}$, are arranged in decreasing order: $\mu_0 \geq \mu_1 \geq \mu_2 \geq \cdots$. We then say that $T$ is an *infinitesimal of order $\alpha$* if

$$\mu_k(T) = O(k^{-\alpha}) \quad \text{as } k \to \infty.$$

Notice that infinitesimals of *first order* have singular values that form a *logarithmically divergent series*:

$$\mu_k(T) = O\left(\frac{1}{k}\right) \implies \sigma_N(T) := \sum_{k<N} \mu_k(T) = O(\log N) \quad \text{as } N \to \infty. \quad (3.3)$$

The dimension axiom then entails that there is a positive integer $n$ for which the singular values of $D^{-n}$ form a logarithmically divergent series. The coefficient of logarithmic divergence will be denote by $\int |D|^{-n}$, where $\int$ denotes the *noncommutative integral*; we shall say more about it in Chapter 5.

*We allow $n = 0$ as a possible metric dimension*, under two circumstances: either (a) the singular values of $|D|^{-1}$ go to zero exponentially fast: see [30], [62] for examples of that; or (b) the algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}$ are finite-dimensional, so that $D$ is just a hermitian matrix. These 'finite spectral triples' have been classified and studied in [117], [136], [165].

**Example.** Let us compute the dimension of the sphere $\mathbb{S}^2$ from its fundamental spectral triple. From the spectrum of $\not{D}$ we get the eigenvalues of the positive operator $\not{D}^{-2}$:

$$\text{sp}(\not{D}^{-2}) = \left\{\left(l + \tfrac{1}{2}\right)^{-2} : l \in \mathbb{N} + \tfrac{1}{2}\right\} = \{k^{-2} : k = 1, 2, 3, \ldots\}$$

where the eigenvalue $k^{-2}$ has multiplicity $4k = 2(2l + 1)$. For $N = 2M(M + 1)$, we get

$$\sigma_N(\not{D}^{-2}) = \sum_{k<M} \frac{4k}{k^2} \sim 4 \log M \sim 2 \log N \quad \text{as } N \to \infty,$$

so that $\not{D}^{-1}$ is an infinitesimal of order $\frac{1}{2}$ and therefore the dimension is 2 (surprise!). Also, the coefficient of logarithmic divergence is

$$\int \not{D}^{-2} = \lim_{N \to \infty} \frac{\sigma_N(\not{D}^{-2})}{\log N} = 2.$$

As we shall see later on, this coefficient is $1/2\pi$ times the area for any 2-dimensional surface, so the area of the sphere is hereby computed to be $4\pi$.

## 3.3  The first-order condition

**Axiom 2** (Order one). *For all $a, b \in \mathcal{A}$, the following commutation relation holds:*

$$[[D, a], Jb^*J^\dagger] = 0. \tag{3.4}$$

This could be rewritten as $[[D, a], b^\circ] = 0$ or as $[[D, \pi(a)], \pi^\circ(b)] = 0$. Using (3.2) and the Jacobi identity, we see that this condition is symmetric in the representations $\pi$ and $\pi^\circ$, since

$$[a, [D, b^\circ]] = [[a, D], b^\circ] + [D, [a, b^\circ]] = -[[D, a], b^\circ] = 0.$$

In the commutative case, the condition (3.4) expresses the fact that the Dirac operator is a first-order differential operator:

$$[[\slashed{D}, a], Jb^*J^\dagger] = [[\slashed{D}, a], b] = -i\,[\gamma(da), b] = 0.$$

(Contrast this with a second-order operator like a Laplacian, that satisfies $[[\Delta, a], b] = 2g^{-1}(da, db)$, generally nonzero [12].)

We can interpret (3.4) as saying that the operators $\pi^\circ(b)$ commute with the sub-algebra of operators on $\mathcal{H}$ generated by all $\pi(a)$ and $[D, \pi(a)]$. This gives rise to a linear representation of the tensor product of several copies of $\mathcal{A}$:

$$\pi_D(a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) := \pi(a)\,[D, \pi(a_1)]\,[D, \pi(a_2)]\dots[D, \pi(a_n)],$$

or, more simply, $a\,[D, a_1]\,[D, a_2]\dots[D, a_n]$. In view of the order one condition, we could even replace the first entry $a \in \mathcal{A}$ by $a \otimes b^\circ \in \mathcal{A} \otimes \mathcal{A}^\circ$, writing

$$\begin{aligned}\pi_D((a \otimes b^\circ) &\otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) \\ &:= \pi(a)\pi^\circ(b)\,[D, \pi(a_1)]\,[D, \pi(a_2)]\dots[D, \pi(a_n)],\end{aligned} \tag{3.5}$$

Now $C_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ) := (\mathcal{A} \otimes \mathcal{A}^\circ) \otimes \mathcal{A}^{\otimes n}$ is a bimodule over the algebra $\mathcal{A}$, and this recipe represents it by operators on $\mathcal{H}$. Its elements are called *Hochschild n-chains* with coefficients in the $\mathcal{A}$-bimodule $\mathcal{A} \otimes \mathcal{A}^\circ$.

## 3.4  Smoothness of the algebra

**Axiom 3** (Regularity). *For any $a \in \mathcal{A}$, $[D, a]$ is a bounded operator on $\mathcal{H}$, and both $a$ and $[D, a]$ belong to the domain of smoothness $\bigcap_{k=1}^{\infty} \mathrm{Dom}(\delta^k)$ of the derivation $\delta$ on $\mathcal{L}(\mathcal{H})$ given by $\delta(T) := [|D|, T]$.*

In the commutative case, where $[\slashed{D}, a] = -i\,\gamma(da)$, this condition amounts to saying that $a$ has derivatives of all orders, i.e., that $\mathcal{A} \subseteq C^\infty(M)$. This can be proved

with pseudodifferential calculus, since the principal symbol of the modulus of the Dirac operator is $\sigma_{|D|}(x, \xi) = |\xi| \, 1$. One finds that all multiplication operators in $\bigcap_{k=1}^{\infty} \mathrm{Dom}(\delta^k)$ are multiplications by smooth functions.

Assuming regularity, one can confer a locally convex topology on $\mathcal{A}$ with the semi-norms $a \mapsto \|\delta^k(a)\|$ and $a \mapsto \|\delta^k([D, a])\|$. The completion of $\mathcal{A}$ in this topology is a Fréchet pre-$C^*$-algebra, and Axiom 3 still holds when $\mathcal{A}$ is replaced by its completion [177, Lemma 16]. Also, this topology coincides with the usual one on $C^\infty(M)$, in the commutative case [177, Prop. 20]. Therefore, we may assume that the algebra of a regular spectral triple is a pre-$C^*$-algebra.

## 3.5 Hochschild cycles and orientation

**Axiom 4** (Orientability). *There is a Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$ whose representative on $\mathcal{H}$ is*

$$\pi_D(c) = \begin{cases} \Gamma, & \text{if } n \text{ is even}, \\ 1, & \text{if } n \text{ is odd}. \end{cases}$$

Here $c$ is a Hochschild $n$-chain as defined above, and $\pi(c)$ is given by (3.5). We say $c$ is a *cycle* if its boundary is zero, where the Hochschild boundary $b(x)$ of the $n$-chain $x = m \otimes a_1 \otimes \cdots \otimes a_n$, with $m \in \mathcal{A} \otimes \mathcal{A}^\circ$, is

$$b(x) := ma_1 \otimes a_2 \otimes \cdots \otimes a_n - m \otimes a_1 a_2 \otimes \cdots \otimes a_n + \cdots$$
$$+ (-1)^{n-1} m \otimes a_1 \otimes \cdots \otimes a_{n-1} a_n + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

This satisfies $b^2 = 0$ and thus makes $C_\bullet(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$ a chain complex, whose homology is the Hochschild homology $H_\bullet(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$. For the full story, see [155]. Notice that if $m = a \otimes b^\circ$, then, by telescoping,

$$\pi_D(b(x)) = (-1)^{n-1}\big(ab^\circ [D, a_1] \ldots [D, a_{n-1}]a_n - a_n ab^\circ [D, a_1] \ldots [D, a_{n-1}]\big).$$

This Hochschild cycle $c$ is the algebraic equivalent of a *volume form* on our non-commutative manifold. To see that, let us look briefly at the commutative case, where we may replace $\mathcal{A} \otimes \mathcal{A}^\circ$ simply by $\mathcal{A}$. A differential form in $\mathcal{A}^k(M)$ is a sum of terms $a_0 \, da_1 \wedge \cdots \wedge da_k$, but in the noncommutative case the skewsymmetry of the wedge product is lost, so we replace such a form with

$$c' := \sum_\sigma (-1)^\sigma a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} \tag{3.6}$$

(sum over $n$-permutations) in $\mathcal{A}^{\otimes(n+1)} = C_n(\mathcal{A}, \mathcal{A})$. Then $bc' = 0$ by cancellation since $\mathcal{A}$ is commutative; for instance:

$$b(a \otimes a' \otimes a'' - a \otimes a'' \otimes a') = (aa' - a'a) \otimes a'' - a \otimes (a'a'' - a''a') + (a''a - aa'') \otimes a'.$$

In the case $\mathcal{A} = C^\infty(M)$, chains are represented by Clifford products: $\pi_{\not{D}}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-i)^n a_0 \gamma(da_1) \dots \gamma(da_n)$. The Riemannian volume form on $M$ can be written as $\Omega = \theta^1 \wedge \cdots \wedge \theta^n$ where $\{\theta^1, \dots, \theta^n\}$ is an oriented orthonormal basis of 1-forms. The cycle $c$ corresponding to $i^{\lfloor (n+1)/2 \rfloor} \Omega$ is represented by $\pi_{\not{D}}(c) = (-i)^{\lfloor n/2 \rfloor} \gamma(\theta^1) \dots \gamma(\theta^n) = \gamma(\chi)$, where $\chi$ is the chirality element – the physicists' '$\gamma_5$' – of the Clifford algebra [104, Sec. 5.1], represented by the grading operator on spinors if $n$ is even, and by 1 if $n$ is odd.

## 3.6 Finiteness of the $K$-cycle

**Axiom 5** (Finiteness). *The space of smooth vectors $\mathcal{H}^\infty := \bigcap_{k=1}^\infty \mathrm{Dom}(D^k)$ is a finitely generated projective left $\mathcal{A}$-module with a Hermitian pairing $(\cdot \mid \cdot)$ given by*

$$\fint (\xi \mid \eta) \, ds^n := \langle \eta \mid \xi \rangle. \tag{3.7}$$

The representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and the regularity axiom already make $\mathcal{H}^\infty$ a left $\mathcal{A}$-module. The definition (1.1) of pairings for right $\mathcal{A}$-modules is easily adapted to the case of left $\mathcal{A}$-modules, provided that the pairing is now linear in the *first* argument; thus, for instance, $(a\xi \mid \eta) = a (\xi \mid \eta)$, so that the previous equation entails

$$\fint a \, (\xi \mid \eta) \, ds^n = \langle \eta \mid a\xi \rangle.$$

To see how this defines a Hermitian pairing implicitly, notice that if $a \in \mathcal{A}$ then $a \, ds^n = a \, |D|^{-n}$ is an infinitesimal of first order, so that the left hand side is defined provided $(\xi \mid \eta) \in \mathcal{A}$. As a finitely generated projective left $\mathcal{A}$-module, $\mathcal{H}^\infty \simeq \mathcal{A}^m p$ with $p = p^2 = p^*$ in some $M_m(\mathcal{A})$. In the commutative case, Connes's trace theorem (see below) shows that $(\xi \mid \eta)$ is just the hermitian product of spinors given by the metric on the spinor bundle.

A point to notice is that

$$\fint a \, (\xi \mid \eta) \, ds^n = \langle \eta \mid a\xi \rangle = \langle a^*\eta \mid \xi \rangle = \fint (\xi \mid a^*\eta) \, ds^n = \fint (\xi \mid \eta) \, a \, ds^n,$$

so this axiom implies that $\fint (\cdot) |D|^{-n}$ defines a finite *trace* on the algebra $\mathcal{A}$. As shown by Cipriani *et al.* [31, Prop. 1.6], the tracial property follows from the regularity axiom; see also [104, Sec. 10.3].

The existence of a finite trace on $\mathcal{A}$ implies that the von Neumann algebra $\mathcal{A}''$ generated by $\mathcal{A}$ also has a finite normal trace, so it cannot have components of types $I_\infty$, $II_\infty$ or $III$ [121, Sec. 8.5]. The finiteness and regularity axioms entail [40] that $\mathcal{A} = \{ T \in \mathcal{A}'' : T \in \bigcap_{k=1}^\infty \mathrm{Dom}(\delta^k) \}$.

## 3.7 Poincaré duality and $K$-theory

**Axiom 6** (Poincaré duality). *The Fredholm index of the operator D yields a nondegenerate intersection form on the K-theory ring of the algebra $\mathcal{A} \otimes \mathcal{A}^\circ$.*

On a compact oriented $n$-dimensional manifold $M$, Poincaré duality is usually formulated [65] as an isomorphism of cohomology (in degree $k$) with homology (in degree $n - k$), or equivalently as a nondegenerate bilinear pairing on the cohomology ring $H^\bullet(M)$. (For noncompact manifolds, the pairing is between the ordinary and the compactly supported cohomologies.) If $\alpha \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^{n-k}(M)$ are closed forms, integration over $M$ pairs them by

$$(\alpha, \eta) \mapsto \int_M \alpha \wedge \eta;$$

since the right hand side depends only on the cohomology classes of $\alpha$ and $\beta$ (it vanishes if either $\alpha$ or $\beta$ is exact), it gives a bilinear map $H^k(M) \times H^{n-k}(M) \to \mathbb{C}$. Now each $\mathcal{A}^k(M)$ carries a scalar product $(\cdot \mid \cdot)$ induced by the metric and orientation on $M$, given by

$$\alpha \wedge \star\beta =: \varepsilon_k \, (\alpha \mid \beta) \, \Omega \quad \text{for } \alpha, \beta \in \mathcal{A}^k(M),$$

where $\varepsilon_k = \pm 1$ or $\pm i$ and $\Omega$ is the volume form on $M$: compare (2.14). This pairing is nondegenerate since

$$\int_M \alpha \wedge (\varepsilon_k^{-1} \star\alpha) = \int_M (\alpha \mid \alpha) \, \Omega > 0 \quad \text{for } \alpha \neq 0.$$

In view of the isomorphism between $K^\bullet(M) \otimes_\mathbb{Z} \mathbb{Q}$ and $H^\bullet(M, \mathbb{Q})$ given by the Chern character, one could hope to reformulate this as a canonical pairing on the $K$-theory ring. This can be done if $M$ is a spin$^c$ manifold; the role of the orientation $[\Omega]$ in cohomology is replaced by the $K$-orientation, so that the corresponding pairing of $K$-rings is mediated by the Dirac operator: see [37, IV.1.$\gamma$] or [34] for a discussion of how this pairing arises. We leave aside the translation from $K$-theory to cohomology (by no means a short story) and explain briefly how the intersection form may be computed in the $K$-context.

**$K$-theory of algebras.** There are two abelian groups, $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$, associated to a Fréchet pre-$C^*$-algebra $\mathcal{A}$, as follows [15], [189], [221]. The group $K_0(\mathcal{A})$ gives a rough classification of finitely generated projective modules over $\mathcal{A}$. If $M_\infty(\mathbb{C})$ is the algebra of compact operators of finite rank, then $M_\infty(\mathcal{A}) = \mathcal{A} \otimes M_\infty(\mathbb{C})$ is a dense $*$-subalgebra of $A \otimes \mathcal{K}$. Two projectors in $M_\infty(\mathcal{A})$ have a direct sum $p \oplus q = \left( \begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix} \right)$. Two such projectors $p$ and $q$ are equivalent if $p = v^*v$ and $q = vv^*$ for some $v \in M_\infty(\mathcal{A})$. Adding the equivalence classes by $[p] + [q] := [p \oplus q]$, we get a semigroup, and

one can always construct a corresponding group of formal differences $[p] - [q]$: this is $K_0(\mathcal{A})$.

The other group $K_1(\mathcal{A})$ is generated by classes of *unitary* matrices over $\mathcal{A}$. We nest the unitary groups of various sizes by identifying $u \in U_m(\mathcal{A})$ with $u \oplus 1_k \in U_{m+k}(\mathcal{A})$, and call $u, v$ equivalent if there is a continuous path from $u$ to $v$ in $U_\infty(\mathcal{A}) := \bigcup_{m \geq 1} U_m(\mathcal{A})$. The group $K_1(\mathcal{A})$ consists of the equivalence classes of $U_\infty(\mathcal{A}^+)$ under this relation. It turns out that $[uv] = [u \oplus v] = [v \oplus u] = [vu]$, so that $K_1(\mathcal{A})$ is abelian. This is the standard definition of $K_1$ for a $C^*$-algebra $A$ [189]; to define it thus for a pre-$C^*$-algebra $\mathcal{A}$ one needs some arguments from homotopy theory that require $\mathcal{A}$ to be also Fréchet [15]. (What we have defined here is a 'topological' $K$-theory: there is a somewhat different 'algebraic' $K$-theory, defined for wider classes of algebras. The $K_0$-groups in both theories coincide, but the $K_1$-groups generally do not. See [17], [190] for the algebraic $K$-theory.)

Both groups are homotopy invariant: if $\{ p_t : 0 \leq t \leq 1 \}$ is a homotopy of projectors in $M_\infty(\mathcal{A})$ and if $\{ u_t : 0 \leq t \leq 1 \}$ is a homotopy in $U_\infty(\mathcal{A})$, then $[p_0] = [p_1]$ in $K_0(\mathcal{A})$ and $[u_0] = [u_1]$ in $K_1(\mathcal{A})$. In the commutative case, we get $K_j(C^\infty(M)) = K^j(M)$ for $j = 0, 1$.

When $\mathcal{A}$ is represented on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, any odd selfadjoint Fredholm operator $D$ on $\mathcal{H}$ defines an index map $\phi_D \colon K_0(\mathcal{A}) \to \mathbb{Z}$, as follows. Denote by $a \mapsto a^+ \oplus a^-$ the representation of $M_m(\mathcal{A})$ on $\mathcal{H}_m^+ \oplus \mathcal{H}_m^- = \mathcal{H}_m = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ ($m$ times); write $D_m := D \oplus \cdots \oplus D$, acting on $\mathcal{H}_m$. Then $p^- D_m p^+$ is a Fredholm operator from $\mathcal{H}_m^+$ to $\mathcal{H}_m^-$, whose index depends only on the class $[p]$ in $K_0(\mathcal{A})$. We define

$$\phi_D\big([p]\big) := \mathrm{index}(p^- D_m p^+).$$

On the other hand, when $\mathcal{A}$ is represented on an ungraded Hilbert space $\mathcal{H}$, a selfadjoint Fredholm operator $D$ on $\mathcal{H}$ defines an index map $\phi_D \colon K_1(\mathcal{A}) \to \mathbb{Z}$. Let $\mathcal{H}_m^>$ be the range of the spectral projector $P_m^> = P_{(0,\infty)}$ determined by the positive part of the spectrum of $D_m$. Then if $u \in U_m(\mathcal{A})$, $P_m^> u P_m^>$ is a Fredholm operator on $\mathcal{H}_m^>$, whose index depends only on the class $[u]$ in $K_1(\mathcal{A})$. We define

$$\phi_D\big([u]\big) := \mathrm{index}(P_m^> u P_m^>).$$

One can work with $K_0$ alone since there is a natural 'suspension' isomorphism between $K_1(A)$ and $K_0(A \otimes C_0(\mathbb{R}))$ for any $C^*$-algebra $A$ [189, Thm. 10.1.3].

**The intersection form.** Coming back now to the spectral triple under discussion, we define a pairing on $K_\bullet(\mathcal{A}) = K_0(\mathcal{A}) \oplus K_1(\mathcal{A})$ as follows. The commuting representations $\pi, \pi^\circ$ determine a representation of the algebra $\mathcal{A} \otimes \mathcal{A}^\circ$ on $\mathcal{H}$ by

$$a \otimes b^\circ \longmapsto a J b^* J^\dagger = J b^* J^\dagger a.$$

If $[p], [q] \in K_0(\mathcal{A})$, then $[p \otimes q^\circ] \in K_0(\mathcal{A} \otimes \mathcal{A}^\circ)$. The *intersection form* for $D$ is

$$\langle [p], [q] \rangle := \phi_D([p \otimes q^\circ]).$$

Combined with the suspension isomorphism, we get three other pairing maps $K_i(\mathcal{A}) \times K_j(\mathcal{A}^\circ) \to K_{i+j}(\mathcal{A} \otimes \mathcal{A}^\circ) \to \mathbb{Z}$, where the second arrow is $\phi_D$.

Poincaré duality is the assertion that this pairing on $K_\bullet(\mathcal{A})$ is nondegenerate. For the case of the finite-dimensional algebra $\mathcal{A} = \mathbb{C} \oplus \mathcal{H} \oplus M_3(\mathbb{C})$ that acts on the space of fermions of the Standard Model [192], the intersection form has been computed in [39]; see also [159, Sec. 6.2]. For more general finite-dimensional algebras, see [136], [165]: in that context, Poincaré duality is used to rule out several plausible alternatives to the Standard Model.

## 3.8 The real structure

**Axiom 7** (Reality). *There is an antilinear isometry $J \colon \mathcal{H} \to \mathcal{H}$ such that the representation $\pi^\circ(b) := J\pi(b^*)J^\dagger$ commutes with $\pi(\mathcal{A})$ and satisfies*

$$J^2 = \pm 1, \quad JD = \pm DJ, \quad J\Gamma = \pm \Gamma J, \tag{3.8}$$

*where the signs are given by the following tables:*

n even:

| $n$ mod 8 | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $J^2 = \pm 1$ | + | − | − | + |
| $JD = \pm DJ$ | + | + | + | + |
| $J\Gamma = \pm \Gamma J$ | + | − | + | − |

n odd:

| $n$ mod 8 | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $J^2 = \pm 1$ | + | − | − | + |
| $JD = \pm DJ$ | − | + | − | + |

These tables, with their periodicity in steps of 8, arise from the structure of real Clifford algebra representations that underlie real $K$-theory. See [4], [149] for the algebraic foundation of this real Bott periodicity. In the commutative case of Riemannian spin manifolds, the charge conjugation operator $J$ on spinors satisfies the foregoing sign rules. Thus, for instance, $J^2 = -1$ for Dirac spinors over 4-dimensional spaces with Euclidean signature. Notice also that the signs for dimension 2 are those we have used in the example of the Riemann sphere. A detailed construction of this sign table is given in [104, Sec. 9.5].

**The Tomita involution.** We should explain where the antilinear operator $J$ comes from, in the noncommutative case. The bicommutant $\mathcal{A}''$ of the involutive algebra $\mathcal{A}$,

or more precisely of $\pi(\mathcal{A})$, is a weakly closed algebra of operators on $\mathcal{H}$, i.e., a von Neumann algebra. Let us assume that $\mathcal{H}$ contains a *cyclic and separating vector* $\xi_0$ for $\mathcal{A}''$, that is, a vector such that (i) $\mathcal{A}''\xi_0$ is a dense subspace of $\mathcal{H}$ (cyclicity) and (ii) $a\xi_0 = 0$ in $\mathcal{H}$ only if $a = 0$ in $\mathcal{A}''$ (separation). A basic result of operator algebras, Tomita's theorem [121], [207], says that the antilinear mapping $a\xi_0 \mapsto a^*\xi_0$ extends to a closed antilinear operator $S$ on $\mathcal{H}$, whose polar decomposition $S = J\Delta^{1/2}$ determines an antiunitary operator $J\colon \mathcal{H} \to \mathcal{H}$ with $J^2 = 1$ such that $a \mapsto Ja^*J^\dagger$ is an isomorphism from $\mathcal{A}''$ onto its commutant $\mathcal{A}'$. (Since these commuting operator algebras are isomorphic, the space $\mathcal{H}$ can be neither too small nor too large; this is what the cyclic and separating vector ensures.)

When the state of $\mathcal{A}''$ given by $a \mapsto \langle \xi_0 \mid a\xi_0 \rangle$ is a trace, the operator $\Delta = S^*S$ is just 1, and so the mapping $S$ is $J$ itself. From (3.7), the trace $\oint(\cdot)\,|D|^{-n}$ on $\mathcal{A}$ gives rise to a tracial vector state on $\mathcal{A}''$. Thus the Tomita theorem already provides us with an antiunitary operator $J$ satisfying $[a, Jb^*J^\dagger] = 0$; we shall see in Chapter 4 how to modify it to obtain $J^2 = -1$ when that is required.

We sum up our discussion with the basic definition.

**Definition 7.** A *noncommutative spin geometry* is a real spectral triple $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$ or $(\mathcal{A}, \mathcal{H}, D; J)$, according as its dimension is even or odd, that satisfies the seven axioms set out above.

Riemannian spin manifolds provide the commutative examples. It is not hard to manufacture noncommutative examples with finite-dimensional matrix algebras [136], [165]; these are zero-dimensional geometries in the sense of Axiom 1. In the next chapter we study a more elaborate noncommutative example which, like the Riemann sphere, has dimension two.

# Geometries on the noncommutative torus

We turn now to an algebra that is not commutative, in order to see how the algebraic formulation of spin geometries allows us to go beyond Riemannian spin manifolds. Moving from commutative to noncommutative algebras is the mathematical point of departure for quantum mechanics. As was already made clear by von Neumann in his 1931 study of the canonical commutation relations [163], the Schrödinger representation arises by replacing convolution of functions of two real variables by a noncommutative 'twisted convolution' [126]. This was reinterpreted by Moyal [162] as a variant of the ordinary product of functions on phase space, related to Weyl's method of quantization.

We begin, then, with Weyl quantization. We wish to quantize a compact phase space, since our formalism so far relies heavily on compactness, e.g., by demanding that the length element $ds = |D|^{-1}$ be a compact operator. The Riemann sphere would seem to be a good candidate, since several studies exist of its quantization both in the Moyal framework [214] and from the noncommutative geometry point of view [105], [157], [174]. However, these involve an approximating sequence of algebras rather a single algebra and so do not define a solitary spectral triple.

We turn instead to the torus $\mathbb{T}^2$ (with the flat metric). Via the Gelfand cofunctor, this is determined by an algebra of *doubly periodic* functions on $\mathbb{R}^2$ (or on $\mathbb{C}$). If $\omega_1, \omega_2$ are the periods, with ratio $\tau := \omega_2/\omega_1$ in the upper half plane $\mathbb{C}_+$, one identifies $\mathbb{T}^2$ with the quotient space $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$. These 'complex tori' are homeomorphic but are not equivalent as complex manifolds. In fact, if one chooses to study these tori through the algebras of *meromorphic* functions with the required double periodicity, the resulting elliptic curves $E_\tau$ are classified by the orbit of $\tau$ under the action $\tau \mapsto (a\tau+b)/(c\tau+d)$ of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathbb{C}_+$.

## 4.1 Algebras of Weyl operators

Our starting point is the canonical commutation relations of quantum mechanics on a one-dimensional configuration space $\mathbb{R}$. In Weyl form [63], [77], these are represented by a family of unitary operators on $L^2(\mathbb{R})$:

$$W_\theta(a, b)\psi : t \longmapsto e^{-\pi i \theta ab}\, e^{2\pi i \theta bt}\, \psi(t - a) \quad \text{for } a, b \in \mathbb{R}. \qquad (4.1)$$

Here $\theta$ is a nonzero real parameter; the reader is invited to think of $\theta$ as $1/\hbar$, the reciprocal of the Planck constant. The linear space generated by $\{W_\theta(a, b) : a, b \in \mathbb{R}\}$

is an involutive algebra, wherein

$$[W_\theta(a, b), W_\theta(c, d)] = -2i \sin\big(\pi\theta(ad - bc)\big) W_\theta(a + c, b + d),$$

so that $W_\theta(a, b)$ and $W_\theta(c, d)$ commute if and only if

$$\theta(ad - bc) \in \mathbb{Z}. \tag{4.2}$$

Moreover, the unitary operators $U_\theta := W_\theta(1, 0)$ and $V_\theta := W_\theta(0, 1)$ obey the commutation relation

$$V_\theta U_\theta = e^{2\pi i\theta} U_\theta V_\theta.$$

The full set of operators $\{W_\theta(a, b) : a, b \in \mathbb{R}\}$ acts irreducibly on $L^2(\mathbb{R})$, but by restricting to *integral* parameters we get reducible actions. We examine first the von Neumann algebra

$$\mathcal{N}_\theta := \{W_\theta(m, n) : m, n \in \mathbb{Z}\}'',$$

whose commutant can be shown to be $\mathcal{N}_\theta' := \{W_\theta(r/\theta, s/\theta) : r, s \in \mathbb{Z}\}''$ [180]. The change of scale given by $(R_\theta\psi)(t) := \theta^{-1/2}\psi(t/\theta)$ transforms these operators to

$$R_\theta W_\theta(r/\theta, s/\theta) R_\theta^{-1} = W_{1/\theta}(r, s),$$

from which we conclude that $\mathcal{N}_\theta' \simeq \mathcal{N}_{1/\theta}$.

The *centre* $\mathcal{Z}(\mathcal{N}_\theta) = \mathcal{N}_\theta \cap \mathcal{N}_\theta'$ depends sensitively on the value of $\theta$. Suppose $\theta$ is rational, $\theta = p/q$ where $p, q$ are integers with $\gcd(p, q) = 1$. Then

$$\mathcal{Z}(\mathcal{N}_{p/q}) = \{W_{p/q}(qm, qn) : m, n \in \mathbb{Z}\}''.$$

This commutative von Neumann algebra is generated by the translation $\psi(t) \mapsto \psi(t - q)$ and the multiplication $\psi(t) \mapsto e^{2\pi i p t} \psi(t)$, and can be identified to the multiplication operators on periodic functions (of period $q$); thus $\mathcal{Z}(\mathcal{N}_{p/q}) \simeq L^\infty(\mathbb{S}^1)$.

On the other hand, if $\theta$ is *irrational*, then $\mathcal{N}_\theta$ is a *factor*, i.e., it has trivial centre $\mathcal{Z}(\mathcal{N}_\theta) = \mathbb{C}\,1$.

If we try to apply the Tomita theory at this stage, we get an unpleasant surprise [77]: $\mathcal{N}_\theta$ has a cyclic vector iff $|\theta| \leq 1$, whereas it has a separating vector iff $|\theta| \geq 1$. This tells us that (if $\theta \neq \pm1$) the space $L^2(\mathbb{R})$ is not the right Hilbert space, and we should represent the algebra generated by the $W_\theta(m, n)$ somewhere else.

To find the good Hilbert space, we must first observe that there is a faithful normal trace on $\mathcal{N}_\theta$ determined by

$$\tau_0(W_\theta(m, n)) := \begin{cases} 1, & \text{if } (m, n) = (0, 0), \\ 0, & \text{otherwise.} \end{cases}$$

The GNS Hilbert space associated to this trace is what we need. Before constructing it, notice that, for $\theta$ irrational, $\mathcal{N}_\theta$ is a factor with a finite normal trace; and it turns out that its relative dimension function [37, V.1.$\gamma$] has range $[0, 1]$, so that $\mathcal{N}_\theta$ is a factor of type $\mathrm{II}_1$ in the Murray–von Neumann classification.

## 4.2  The algebra of the noncommutative torus

We leave the 'measure-theoretic' level of von Neumann algebras and focus on the pre-$C^*$-algebra $\mathcal{A}_\theta$ generated by the operators $W_\theta(m, n)$. It is better to start afresh with a more abstract approach. We redefine $\mathcal{A}_\theta$ as follows.

**Definition 8.**  For a fixed real number $\theta$, let $A_\theta$ be the unital $C^*$-algebra generated by two elements $u$, $v$ subject only to the relations $uu^* = u^*u = 1$, $vv^* = v^*v = 1$, and

$$vu = \lambda\, uv \quad \text{where } \lambda := e^{2\pi i\theta}. \tag{4.3}$$

Let $\mathcal{S}(\mathbb{Z}^2)$ denote the double sequences $a = \{a_{rs}\}$ that are *rapidly decreasing* in the sense that

$$p_k(a)^2 = \sup_{r,s\in\mathbb{Z}} (1 + r^2 + s^2)^k\, |a_{rs}|^2 < \infty \quad \text{for all } k \in \mathbb{N}.$$

The *noncommutative torus algebra* $\mathcal{A}_\theta$ is defined as

$$\mathcal{A}_\theta := \big\{ a = \textstyle\sum_{r,s} a_{rs}\, u^r v^s : \underline{a} \in \mathcal{S}(\mathbb{Z}^2) \big\}. \tag{4.4}$$

With the topology given by the seminorms $p_k$, $\mathcal{A}_\theta$ is a Fréchet pre-$C^*$-algebra that is dense in $A_\theta$.

The product and involution in $\mathcal{A}_\theta$ are computable from (4.3):

$$ab = \sum_{r,s} a_{r-n,m}\, \lambda^{mn}\, b_{n,s-m}\, u^r v^s, \quad a^* = \sum_{r,s} \lambda^{rs}\, \bar{a}_{-r,-s}\, u^r v^s. \tag{4.5}$$

The Weyl operators $U_\theta$, $V_\theta$ are unitary and obey (4.3); thus they generate a faithful concrete representation of this pre-$C^*$-algebra. On the other hand, the relation (4.3) alone generates the algebra $\mathbb{C}_\lambda[u, v]$ known as the Manin $q$-plane [125] for $q = \lambda \in \mathbb{T}$. Here we also require unitarity of the generators, so that $\mathcal{A}_\theta$ is (a completion of) a quotient algebra of this $q$-plane.

There is another representation of $A_\theta$ on $L^2(\mathbb{T})$: the multiplication operator $U$ and the rotation operator $V$, given by $(U\psi)(z) := z\psi(z)$ and $(V\psi)(z) := \psi(\lambda z)$, satisfy (4.3). Here, $U$ generates the $C^*$-algebra $C(\mathbb{T})$ and conjugation by $V$ gives an automorphism $\alpha$ of $C(\mathbb{T})$. Under such circumstances, the $C^*$-algebra generated by $C(\mathbb{T})$ and the unitary operator $V$ is called the *crossed product* of $C(\mathbb{T})$ by $\alpha$ (or rather, by the automorphism group $\{\alpha^n : n \in \mathbb{Z}\}$). In symbols,

$$A_\theta \simeq C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}.$$

If $\theta$ is irrational, the corresponding action by the rotation angle $2\pi\theta$ on the circle is ergodic and minimal (all orbits are dense); it is known [173] that the $C^*$-algebra $A_\theta$ is therefore *simple*.

One advantage of using the abstract presentation by (4.3) and (4.4) to define the algebras $\mathcal{A}_\theta$ is that certain *isomorphisms* become evident. First of all, $\mathcal{A}_\theta \simeq \mathcal{A}_{\theta+n}$ for any $n \in \mathbb{Z}$, since $\lambda$ is the same for both. (Note, however, that their representations by Weyl operators, while equivalent, are not identical: indeed, $V_{\theta+n} = e^{2\pi i n t} V_\theta$. Hence the von Neumann algebras $\mathcal{N}_\theta$ and $\mathcal{N}_{\theta+n}$ do not coincide.)

Next, $\mathcal{A}_\theta \simeq \mathcal{A}_{-\theta}$ via the isomorphism determined by $u \mapsto v$, $v \mapsto u$. There are no more isomorphisms among the $\mathcal{A}_\theta$, however: by computing the $K_0$-groups of these algebras, Rieffel [179] showed that the ordered abelian group $\mathbb{Z}+\mathbb{Z}\theta$ is an isomorphism invariant of the $C^*$-algebra $A_\theta$.

Some *automorphisms* of $\mathcal{A}_\theta$ are also easy to find. The map $u \mapsto u^{-1}$, $v \mapsto v^{-1}$ is one such. More generally, if $a, b, c, d \in \mathbb{Z}$, then

$$\sigma(u) := u^a v^b, \quad \sigma(v) := u^c v^d \tag{4.6}$$

yields $\sigma(v)\sigma(u) = \lambda^{ad-bc}\sigma(u)\sigma(v)$, so this map extends to an automorphism of $\mathcal{A}_\theta$ if and only if $ad - bc = 1$, when $\theta$ is irrational.

When $\theta = 0$, we can identify $u, v$ with multiplications by $z_1 = e^{2\pi i \phi_1}$, $z_2 = e^{2\pi i \phi_2}$ on $\mathbb{T}^2$, so that $\mathcal{A}_0 = C^\infty(\mathbb{T}^2)$. The presentation $a = \sum_{r,s} a_{rs} u^r v^s$ is the expansion of a smooth function on the torus in a *Fourier series*:

$$a(\phi_1, \phi_2) = \sum_{r,s \in \mathbb{Z}} a_{rs}\, e^{2\pi i (r\phi_1 + s\phi_2)},$$

and (4.6) gives the hyperbolic automorphisms of the torus. For other rational values $\theta = p/q$ (with $p \in \mathbb{Z}, q \in \mathbb{N}$ having no common factor), it turns out that $\mathcal{A}_{p/q}$ is *Morita-equivalent* to $C^\infty(\mathbb{T}^2)$; for an explicit construction of the equivalence bimodules, we remit to [182]. Indeed, $\mathcal{A}_{p/q}$ is the algebra of smooth sections of a certain bundle of $q \times q$ matrices over $\mathbb{T}^2$, and its centre is isomorphic to $C^\infty(\mathbb{T}^2)$. The algebras $\mathcal{A}_{p/q}$, for $0 \leq p/q < \frac{1}{2}$, are mutually non-isomorphic.

**The normalized trace.** The restriction of the faithful normal trace on $\mathcal{N}_\theta$ is a faithful trace on $A_\theta$ [173]; it is the linear functional $\tau \colon A_\theta \to \mathbb{C}$ given by

$$\tau(a) := a_{00}.$$

Notice that $\tau(a^*a) = \sum_{r,s} |a_{rs}|^2 > 0$ for $a \neq 0$; it satisfies $\tau(1) = 1$ and is a trace, since $\tau(ab) = \tau(ba)$ from (4.5). When $\theta$ is irrational, this normalized trace on $A_\theta$ is unique.

The Weyl operators (4.1) allow us to quantize $\mathcal{A}_0$ in such a way that $a(\phi_1, \phi_2)$ is the symbol of the operator $a \in \mathcal{A}_\theta$. Then it can be proved [63, Thm. 3] that $\tau(a) = \int_{\mathbb{T}^2} a(\phi_1, \phi_2)\, d\phi_1\, d\phi_2$, so that $\tau$ is just the integral of the classical symbol.

The *GNS representation space* $\mathcal{H}_0 = L^2(\mathcal{A}_\theta, \tau)$ may be described as the completion of the vector space $\mathcal{A}_\theta$ in the Hilbert norm

$$\|a\|_2 := \sqrt{\tau(a^*a)}.$$

Since $\tau$ is faithful, the obvious map $\mathcal{A}_\theta \to \mathcal{H}_0$ is injective; to keep the bookkeeping straight, we shall denote by $\underline{a}$ the image in $\mathcal{H}_0$ of $a \in \mathcal{A}_\theta$.

The GNS representation of $\mathcal{A}_\theta$ is just $\pi_0(a) \colon \underline{b} \mapsto \underline{ab}$. Notice that the vector $\underline{1}$ is obviously cyclic and separating, so the Tomita involution is given by

$$J_0(\underline{a}) := \underline{a^*}.$$

The commuting representation $\pi_0^\circ$ is then given by right multiplication:

$$\pi_0^\circ(a)\,\underline{b} := J_0\pi_0(a^*)J_0^\dagger\,\underline{b} = J_0\,\underline{a^*b^*} = \underline{ba}.$$

To build a two-dimensional geometry, we need a $\mathbb{Z}_2$-graded Hilbert space carrying an antilinear involution $J$ that anticommutes with the grading and satisfies $J^2 = -1$. There is a simple device that solves all of these requirements: we simply *double* the GNS Hilbert space by taking $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_0$ and define

$$J := \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}.$$

It remains to introduce the operator $D$. Before doing so, we make a brief excursion into two-dimensional topology.

## 4.3 The skeleton of the noncommutative torus

An ordinary 2-torus may be built up from the following data: (i) a single point; (ii) two lines, adjoined at their ends to the point to make a pair of circles; and (iii) a plane sheet, attached along its borders to the two circles. Thus, the torus has a 'cell decomposition' into a 0-cell, two 1-cells and a 2-cell.

In more technical language, these cells form the *skeleton* of the torus, and are represented by independent homology classes: one in $H_0(\mathbb{T}^2)$, two in $H_1(\mathbb{T}^2)$ and one in $H_2(\mathbb{T}^2)$. The Euler characteristic of the torus is then computed as $1 - 2 + 1 = 0$.

Guided by the Gelfand cofunctor, homology of spaces is replaced by cohomology of algebras; thus the skeleton of the noncommutative torus will consist of a 0-cocycle, two 1-cocycles and a 2-cocycle on the algebra $\mathcal{A}_\theta$. The appropriate theory for that is *cyclic* cohomology [17], [34], [37], [57]. It is a topological theory insofar as it depends only on the algebra $\mathcal{A}_\theta$ and not on the geometries determined by its $K$-cycles.

**Definition 9.** A *cyclic n-cochain* over an algebra $\mathcal{A}$ is an $(n + 1)$-linear form $\phi \colon \mathcal{A}^{n+1} \to \mathbb{C}$ that satisfies the cyclicity condition

$$\phi(a_0, \ldots, a_{n-1}, a_n) = (-1)^n \phi(a_n, a_0, \ldots, a_{n-1}).$$

It is a *cyclic n-cocycle* if $b\phi = 0$, with the coboundary operator $b$ defined by

$$b\phi(a_0, \ldots, a_{n+1}) := \sum_{j=0}^{n} (-1)^j \phi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1})$$
$$+ (-1)^{n+1} \phi(a_{n+1} a_0, a_1, \ldots, a_n).$$

One checks that $b^2 = 0$, so that the cyclic cochains form a complex whose $n$-th cyclic cohomology group is denoted $HC^n(\mathcal{A})$.

The normalized trace $\tau$ is a cyclic 0-cocycle on $\mathcal{A}_\theta$, since

$$b\tau(a, b) := \tau(ab) - \tau(ba) = 0.$$

In fact, a cyclic 0-cocycle is clearly the same thing as a trace. The uniqueness of the normalized trace shows that $HC^0(\mathcal{A}_\theta) = \mathbb{C}[\tau]$ for irrational $\theta$.

**The two basic derivations.** The partial derivatives $\partial/\partial\phi_1$, $\partial/\partial\phi_2$ on the algebra of Fourier series $\mathcal{A}_0 = C^\infty(\mathbb{T}^2)$ can be rewritten as

$$\delta_1(a_{rs} u^r v^s) := 2\pi i r\, a_{rs}\, u^r v^s,$$
$$\delta_2(a_{rs} u^r v^s) := 2\pi i s\, a_{rs}\, u^r v^s. \tag{4.7}$$

These formulas also make sense on $\mathcal{A}_\theta$, and define *derivations* $\delta_1$, $\delta_2$, i.e.,

$$\delta_j(ab) = (\delta_j a)\, b + a\, (\delta_j b), \quad j = 1, 2.$$

They are symmetric, i.e., $(\delta_j a)^* = \delta_j(a^*)$. Each $\delta_j$ extends to an unbounded operator on $A_\theta$ whose smooth domain is exactly $\mathcal{A}_\theta$. Notice that $\tau(\delta_1 a) = \tau(\delta_2 a) = 0$ for all $a$.

The two cyclic 1-cocycles we need are then given by:

$$\psi_1(a, b) := \tau(a\,\delta_1 b), \quad \psi_2(a, b) := \tau(a\,\delta_2 b).$$

These are cocycles because $\delta_1$, $\delta_2$ are derivations:

$$b\psi_j(a, b, c) = \tau(ab\,\delta_j c - a\,\delta_j(bc) + a\,(\delta_j b)c) = 0.$$

It turns out [34] that $HC^1(\mathcal{A}_\theta) = \mathbb{C}[\psi_1] \oplus \mathbb{C}[\psi_2]$.

Next, there is a 2-cocycle obtained by promoting the trace $\tau$ to a cyclic trilinear form:

$$S\tau(a, b, c) := \tau(abc).$$

In fact, one can always promote a cyclic $m$-cocycle on an algebra to a cyclic $(m + 2)$-cocycle by the *periodicity operator* $S$ of Connes [37, III.1.$\beta$]. For instance, for $m = 1$ we get

$$S\psi_j(a, b, c, d) := \psi_j(abc, d) - \psi_j(ab, cd) + \psi_j(a, bcd).$$

However, there is another cyclic 2-cocycle that is not in the range of $S$:

$$\phi(a, b, c) := \frac{1}{2\pi i} \tau(a\,\delta_1 b\,\delta_2 c - a\,\delta_2 b\,\delta_1 c). \qquad (4.8)$$

Its cyclicity $\phi(a, b, c) = \phi(c, a, b)$ and the condition $b\phi = 0$ are easily verified. It turns out that $HC^2(\mathcal{A}_\theta) = \mathbb{C}[S\tau] \oplus \mathbb{C}[\phi]$.

For $m \geq 3$, the cohomology groups are stable under repeated application of $S$, i.e., $HC^m(\mathcal{A}_\theta) = S(HC^{m-2}(\mathcal{A}_\theta)) \simeq \mathbb{C} \oplus \mathbb{C}$. The inductive limit of these groups yields a $\mathbb{Z}_2$-graded ring $HP^0(\mathcal{A}_\theta) \oplus HP^1(\mathcal{A}_\theta)$ called *periodic cyclic cohomology*, with $HP^0$ generated by $[\tau]$ and $[\phi]$, while $HP^1$ is generated by $[\psi_1]$ and $[\psi_2]$. (This ring is the range of the Chern character in noncommutative topology: for that, we refer to [17], [57] or to [37, III].) In this way, the four cyclic cocycles defined above yield a complete description, in algebraic terms, of the homological structure of the noncommutative torus.

The cocycle (4.8) plays an important rôle in the theory of the integer quantum Hall effect [37, IV.6.$\gamma$]: for a comprehensive review, see Bellissard *et al.* [9]. In essence, the Brillouin zone $\mathbb{T}^2$ of a periodic two-dimensional crystal may be replaced in the nonperiodic case by a *noncommutative Brillouin zone* that is none other than the algebra $\mathcal{A}_\theta$ (where $\theta$ is a magnetic flux in units of $h/e$). Provided that a certain parameter $\mu$ (the Fermi level) lies in a gap of the spectrum of the Hamiltonian, with corresponding spectral projector $E_\mu \in \mathcal{A}_\theta$, the Hall conductivity is given by the Kubo formula: $\sigma_H = (e^2/h)\,\phi(E_\mu, E_\mu, E_\mu)$. Here $\phi(E_\mu, E_\mu, E_\mu) = \langle [\phi], [E_\mu] \rangle$, where the latter is an integer-valued pairing of $HP^0(\mathcal{A}_\theta)$ with $K_0(\mathcal{A}_\theta)$, so $\sigma_H$ is predicted to be an integral multiple of $e^2/h$.

## 4.4  A family of spin geometries on the torus

We search now for suitable operators $D$ so that $(\mathcal{A}_\theta, \mathcal{H}, D; \Gamma, J)$ is a 2-dimensional spin geometry. Here $\Gamma$ is the $\mathbb{Z}_2$-grading operator on $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ where $\mathcal{H}^+, \mathcal{H}^-$ are two copies of $L^2(\mathcal{A}_\theta, \tau)$. The known operators on $\mathcal{H}$ are

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi(a) = \begin{pmatrix} \pi_0(a) & 0 \\ 0 & \pi_0(a) \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}.$$

In order that $D$ be selfadjoint and anticommute with $\Gamma$, it must be of the form

$$D = -i \begin{pmatrix} 0 & -\underline{\partial}^\dagger \\ \underline{\partial} & 0 \end{pmatrix},$$

for a suitable closed operator $\underline{\partial}$ on $L^2(\mathcal{A}_\theta, \tau)$.

The first-order axiom demands that $[D, \pi(a)]$ commute with each $\pi^\circ(b) = J\pi(b^*)J^\dagger$, so that $[D, \pi(a)] \in (\pi^\circ(\mathcal{A}_\theta))' = M_2(\pi_0(\mathcal{A}_\theta)'')$, by Tomita's theorem. The regularity axiom and the finiteness property now show that

$$[D, \pi(a)] \in M_2(\pi_0(\mathcal{A}_\theta)'') \cap \bigcap_{k=1}^\infty \mathrm{Dom}(\delta^k) = M_2(\pi_0(\mathcal{A}_\theta)).$$

Since $[D, \pi(a)]$ anticommutes with $\Gamma$, we get (suppressing the $\pi_0$ notation):

$$[D, \pi(a)] = -i \begin{pmatrix} 0 & -[\underline{\partial}^\dagger, a] \\ [\underline{\partial}, a] & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & -\partial^* a \\ \partial a & 0 \end{pmatrix}$$

where $\partial, \partial^*$ are linear maps of $\mathcal{A}_\theta$ into itself. Let us assume also that $\underline{\partial}(\underline{1}) = \underline{\partial}^\dagger(\underline{1}) = 0$. It follows that $\underline{\partial}(\underline{b}) = [\underline{\partial}, b](\underline{1}) = \underline{\partial b}$, and that

$$\tau((\partial a)^* b) = \langle \underline{\partial a} \mid \underline{b} \rangle = \langle \underline{a} \mid \underline{\partial}^\dagger \underline{b} \rangle = \tau(a^* \partial^* b). \tag{4.9}$$

Since $[D, \pi(ab)] = [D, \pi(a)]\pi(b) + \pi(a)[D, \pi(b)]$, the maps $\partial, \partial^*$ are *derivations* of $\mathcal{A}_\theta$. Then $\partial(1) = 0$ and (4.9) show that $\tau \circ \partial^* = 0$, so

$$\tau(b(\partial a)^*) = \tau((\partial^* b) a^*) = -\tau(b\, \partial^* a^*),$$

and therefore $(\partial a)^* = -\partial^* a^*$.

The reality condition $JDJ^\dagger = D$ is equivalent to requiring that $J_0 \underline{\partial} J_0 = -\underline{\partial}^\dagger$ on $\mathcal{H}_0 = L^2(\mathcal{A}_\theta, \tau)$. Since

$$[J_0 \underline{\partial} J_0, a^*] = (\partial a)^* = -\partial^* a^* = -[\underline{\partial}^\dagger, a^*],$$

the operator $J_0 \underline{\partial} J_0 + \underline{\partial}^\dagger$ commutes with $\pi_0(\mathcal{A}_\theta)''$ and kills the cyclic vector $\underline{1}$, so it vanishes identically.

**The derivation $\partial_\tau$.** For concreteness, we take $\partial$ to be a linear combination of the basic derivations $\delta_1, \delta_2$ of (4.7). (This is actually the most general form of $\partial$ when $\theta$ is irrational, outside a nullset of exceptional values [16].) Apart from a scale factor, such a derivation is

$$\partial = \partial_\tau := \delta_1 + \tau \delta_2 \quad \text{with } \tau \in \mathbb{C}. \tag{4.10}$$

(We shall soon see that real values of $\tau$ must be excluded.) Also, since we could replace $\partial_\tau$ by $\tau^{-1}\partial_\tau = \delta_2 + \tau^{-1}\delta_1$, we may assume that $\Im\tau > 0$. It follows from (4.9) that $\partial_\tau^* = -\delta_1 - \bar\tau \delta_2$.

To verify that this putative spin geometry is 2-dimensional, we must check that $|D_\tau|^{-1}$ is an infinitesimal of order $\frac{1}{2}$. Notice that $D_\tau^2 = \underline{\partial}_\tau^\dagger \underline{\partial}_\tau \oplus \underline{\partial}_\tau \underline{\partial}_\tau^\dagger$ and that the vectors $\underline{u^m v^n}$ form an orthonormal basis of eigenvectors for both $\underline{\partial}_\tau^\dagger \underline{\partial}_\tau$ and $\underline{\partial}_\tau \underline{\partial}_\tau^\dagger$. In fact,

$$\partial_\tau^* \partial_\tau (u^m v^n) = \partial_\tau \partial_\tau^* (u^m v^n) = -(\delta_1 + \tau\delta_2)(\delta_1 + \bar\tau\delta_2)(u^m v^n)$$
$$= 4\pi^2 |m + n\tau|^2 u^m v^n.$$

Thus $D_\tau^{-2}$ has a discrete spectrum of eigenvalues $(4\pi^2)^{-1}|m+n\tau|^{-2}$, each with multiplicity 2, and hence is a compact operator. Now the Eisenstein series

$$G_{2k}(\tau) := {\sum_{m,n}}' \frac{1}{(m+n\tau)^{2k}},$$

with primed summation ranging over integer pairs $(m,n) \neq (0,0)$, converges absolutely for $k > 1$ and only conditionally for $k = 1$. We shall see below that $\sum'_{m,n} |m+n\tau|^{-2}$ in fact diverges logarithmically, thereby establishing the two-dimensionality of the geometry.

**The orientation cycle.** In terms of the generators $u = e^{2\pi i\phi_1}$, $v = e^{2\pi i\phi_2}$ of $\mathcal{A}_0$, the volume form on the torus $\mathbb{T}^2$ is $d\phi_1 \wedge d\phi_2 = (2\pi i)^{-2} u^{-1}v^{-1}\, du \wedge dv$, with the corresponding Hochschild cycle:

$$(2\pi i)^{-2}\big(v^{-1}u^{-1} \otimes u \otimes v - u^{-1}v^{-1} \otimes v \otimes u\big). \tag{4.11}$$

This formula still defines a Hochschild 2-cycle over any $\mathcal{A}_\theta$. More generally, if

$$\boldsymbol{c} := m \otimes u \otimes v - n \otimes v \otimes u \in C_2(\mathcal{A}_\theta, \mathcal{A}_\theta \otimes \mathcal{A}_\theta^\circ),$$

then

$$b\boldsymbol{c} = (mu \otimes v - m \otimes uv + vm \otimes u) - (nv \otimes u - n \otimes vu + un \otimes v),$$

so that $\boldsymbol{c}$ is a 2-cycle if and only if $mu = un$, $vm = nv$ and $m = \lambda n$ in $\mathcal{A}_\theta \otimes \mathcal{A}_\theta^\circ$. For instance, since $\mathcal{A}_\theta \simeq \mathcal{A}_\theta \otimes 1^\circ \subset \mathcal{A}_\theta \otimes \mathcal{A}_\theta^\circ$, we can take $m = \alpha v^{-1}u^{-1}$, $n = \alpha u^{-1}v^{-1}$ with a suitable constant $\alpha \in \mathbb{C}$.

The representative $\pi_D(\boldsymbol{c})$ on $\mathcal{H}$ satisfies

$$\alpha^{-1}\pi_D(\boldsymbol{c}) = \pi(v^{-1}u^{-1})\,[D_\tau, \pi(u)]\,[D_\tau, \pi(v)] - \pi(u^{-1}v^{-1})\,[D_\tau, \pi(v)]\,[D_\tau, \pi(u)]$$

$$= \begin{pmatrix} v^{-1}u^{-1}\,\partial_\tau^* u\,\partial_\tau v - u^{-1}v^{-1}\,\partial_\tau^* v\,\partial_\tau u & 0 \\ 0 & v^{-1}u^{-1}\,\partial_\tau u\,\partial_\tau^* v - u^{-1}v^{-1}\,\partial_\tau v\,\partial_\tau^* u \end{pmatrix}.$$

Since

$$\partial_\tau u = 2\pi i u, \quad \partial_\tau^* u = -2\pi i u, \quad \partial_\tau v = 2\pi i\tau v, \quad \partial_\tau^* v = -2\pi i\bar{\tau} v,$$

this reduces to

$$\pi_D(\boldsymbol{c}) = 4\pi^2\alpha \begin{pmatrix} \tau - \bar{\tau} & 0 \\ 0 & \bar{\tau} - \tau \end{pmatrix} = 4\pi^2\alpha(\tau - \bar{\tau})\,\Gamma.$$

Thus the *orientation cycle* is given by

$$\boldsymbol{c} := \frac{1}{4\pi^2(\tau - \bar{\tau})}(v^{-1}u^{-1} \otimes u \otimes v - u^{-1}v^{-1} \otimes v \otimes u). \tag{4.12}$$

This makes sense only if $\tau - \bar{\tau} \neq 0$, i.e., $\tau \notin \mathbb{R}$, which explains why we chose $\Im\tau > 0$. Thus $(\Im\tau)^{-1}$ is a *scale factor* in the metric determined by $D_\tau$. There is a minor difference with the commutative volume form (4.11): since $v^{-1}u^{-1} = \lambda\, u^{-1}v^{-1}$, there is also a phase factor $\lambda = e^{2\pi i\theta}$ in the orientation cycle.

**The area of the noncommutative torus.** To determine the total area, we compute the coefficient of logarithmic divergence of the series given by $\mathrm{sp}(D_\tau^{-2})$. Partially summing over lattice points with $m^2 + n^2 \leq R^2$, we get, with $\tau = s + it$:

$$
\begin{aligned}
\fint D_\tau^{-2} &= \frac{2}{4\pi^2} \lim_{R\to\infty} \frac{1}{2\log R} \sideset{}{'}\sum_{m^2+n^2\leq R^2} \frac{1}{|m+n\tau|^2} \\
&= \frac{1}{4\pi^2} \lim_{R\to\infty} \frac{1}{\log R} \int_1^R \frac{r\,dr}{r^2} \int_{-\pi}^{\pi} \frac{d\theta}{(\cos\theta + s\sin\theta)^2 + t^2\sin^2\theta} \\
&= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \frac{d\theta}{(\cos\theta + s\sin\theta)^2 + t^2\sin^2\theta} \\
&= \frac{1}{4\pi^2}\left(\frac{2\pi}{t}\right) = \frac{i}{\pi(\tau - \bar{\tau})},
\end{aligned}
$$

after an unpleasant contour integration. The area is then $2\pi \fint D^{-2} = 1/\Im\tau$. This area is inversely proportional to the area of the period parallelogram of the elliptic curve $E_\tau$ with periods $\{1, \tau\}$.

A second method of computing the area relies on the existence of a Chern character homomorphism from $K$-homology to cyclic cohomology: see [37, IV.1] and [57]. The general theory suggests that the area will be given by pairing the orientation class $[c] \in H_2(\mathcal{A}_\theta, \mathcal{A}_\theta \otimes \mathcal{A}_\theta^\circ)$ with a class $[\phi] \in HC^2(\mathcal{A}_\theta)$, where $\phi$ is the cyclic cocycle (4.8) that represents the highest level of the skeleton of the torus. This is the image under the Gelfand cofunctor of the familiar process of integrating the volume form over the fundamental cycle of the manifold. At the level of cocycles and cycles, the pairing is defined by $\langle \phi, a_0 \otimes a_1 \otimes a_2 \rangle := \phi(a_0, a_1, a_2)$. Thus

$$
\begin{aligned}
\langle \phi, c \rangle &= \frac{1}{4\pi^2(\tau - \bar{\tau})}\big(\phi(v^{-1}u^{-1}, u, v) - \phi(u^{-1}v^{-1}, v, u)\big) \\
&= \frac{(2\pi i)^{-1}}{4\pi^2(\tau - \bar{\tau})}\tau\big(v^{-1}u^{-1}(\delta_1 u\,\delta_2 v - \delta_2 u\,\delta_1 v) - u^{-1}v^{-1}(\delta_1 v\,\delta_2 u - \delta_2 v\,\delta_1 u)\big) \\
&= \frac{2\pi i}{4\pi^2(\tau - \bar{\tau})}\tau(v^{-1}u^{-1}uv + u^{-1}v^{-1}vu) = \frac{i}{\pi(\tau - \bar{\tau})} = \fint D_\tau^{-2}.
\end{aligned}
$$

The spin geometry $(\mathcal{A}_\theta, \mathcal{H}, D_\tau; \Gamma, J)$ thus depends on two parameters $\theta$ and $\tau$, up to some equivalences that we shall explore in Chapter 7. We may denote it by $\mathbb{T}^2_{\theta,\tau}$, regarding it as the noncommutative space $\mathbb{T}^2_\theta$ gifted with the metric conferred by the Dirac operator $D_\tau$.

**$K$-theory and Poincaré duality.** The noncommutative torus provides an example of a pre-$C^*$-algebra, which is neither commutative nor approximately finite but has an interesting and computable $K$-theory [179]. The classes of the unitaries $u$ and $v$ in $K_1(\mathcal{A}_\theta) = K_1(A_\theta)$ can be paired with the classes of $\psi_1$ and $\psi_2$ in $HC^1(\mathcal{A}_\theta)$; on setting $\langle \psi, w \rangle := (2\pi i)^{-1} \psi(w^{-1}, w)$, we obtain

$$\langle \psi_1, u \rangle = 1, \quad \langle \psi_1, v \rangle = 0, \quad \langle \psi_2, u \rangle = 0, \quad \langle \psi_2, u \rangle = 1.$$

It turns out [73] that $[u]$ and $[v]$ are generators, so that $K_1(\mathcal{A}_\theta) = \mathbb{Z}[u] \oplus \mathbb{Z}[v]$.

To determine $K_0(\mathcal{A}_\theta)$, we seek projectors in $M_k(\mathcal{A}_\theta)$ which are not equivalent to $e := 1 \oplus 0_{k-1}$. When $\theta$ is irrational, such projectors may be found in $\mathcal{A}_\theta$ itself: the *Powers–Rieffel projector* $p \in \mathcal{A}_\theta$ has the characteristic property that $\tau(p) = \theta$. Given this projector, the map $m[e] + n[p] \mapsto m\tau(e) + n\tau(p) = m + n\theta$ defines a map from $K_0(\mathcal{A}_\theta)$ to $\mathbb{Z} + \mathbb{Z}\theta$ which, by a theorem of Pimsner and Voiculescu [170], is an isomorphism of ordered groups.

This $p$ is constructed as follows. Write elements of $\mathcal{A}_\theta$ as $a = \sum_s f_s v^s$, where $f_s = \sum_r a_{rs} u^r$ is a Fourier series expansion of $f_s(t) = \sum_r a_{rs} e^{2\pi i r t}$ in $C^\infty(\mathbb{T})$. Now we look for $p$ of the form $p = gv + f + hv^{-1}$. Since $p^* = p$, the function $f$ is real and $\overline{h(t)} = g(t + \theta)$. Assuming $\frac{1}{2} < \theta < 1$, as we may, we choose $f$ to be a smooth increasing function on $[0, 1 - \theta]$, define $f(t) := 1$ if $t \in [1 - \theta, \theta]$, and $f(t) := 1 - f(t - \theta)$ if $t \in [\theta, 1]$; then let $g$ be the smooth bump function supported in $[\theta, 1]$ given by $g(t) := \sqrt{f(t) - f(t)^2}$ for $\theta \le t \le 1$. One checks that these conditions guarantee $p^2 = p$ (look at the coefficients of $v^2$, $v$ and $1$ in the expansion of $p^2$). Moreover,

$$\tau(p) = a_{00} = \int_0^1 f(t)\, dt = \int_0^{1-\theta} f(t)\, dt + (2\theta - 1) + \int_\theta^1 f(t)\, dt = \theta.$$

The existence of such projectors shows that the topology of the noncommutative torus is very disconnected, in contrast to the ordinary torus $\mathcal{A}_0 = C^\infty(\mathbb{T}^2)$, whose only projectors are 0 and 1.

The torus algebra $C^\infty(\mathbb{T}^2)$ has the same $K$-groups: $K_j(\mathcal{A}_0) = \mathbb{Z} \oplus \mathbb{Z}$ for $j = 1, 2$. This algebra has no Powers–Rieffel projector: the second generator of $K_0(\mathcal{A}_0)$ is obtained by pulling back the Bott projector from $K_0(C^\infty(\mathbb{S}^2))$.

Thus, $K_\bullet(\mathcal{A}_\theta)$ has four generators: $[e]$, $[p]$, $[u]$ and $[v]$. The intersection form is skewsymmetric (this is typical of dimension two) and the nonzero pairings of the generators are [40]:

$$\langle [e], [p] \rangle = -\langle [p], [e] \rangle = 1, \quad \langle [u], [v] \rangle = -\langle [v], [u] \rangle = 1.$$

The matrix of the form is a direct sum of two $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ blocks, so it is nondegenerate, and Poincaré duality holds.

# 5

# The noncommutative integral

One of the striking features of noncommutative geometry is how it ties together many mathematical and physical approaches in a single unifying theme. So far, our discussion of its mathematical underpinnings has been quite 'soft', emphasizing the algebraic character of the overall picture. However, when we examine more closely the interpretation of differential and integral calculus, we see that the theory requires a considerable amount of analysis, of a fairly delicate nature. In this chapter we consider how best to define the noncommutative integral and relate it to conventional integration on manifolds.

In the course of the initial development of noncommutative geometry, integration came first, beginning with Segal's early work with traces on operator algebras [197] and continuing with Connes' work on foliations [33]. The introduction of universal graded differential algebras [34] shifted the emphasis to differential calculus based on derivations, which formed the backdrop for the first applications to particle physics [50], [71]. The pendulum later swung back to integral methods, due to the realization [26], [40], [80], [116] that the Yang–Mills functionals could be obtained in this way. The primary tool for that is the noncommutative integral.

## 5.1 The Dixmier trace on infinitesimals

Early attempts at noncommutative integration [197] used the ordinary trace of Hilbert-space operators as an ersatz integral, where traceclass operators play the role of integrable functions. For example, in Moyal quantization one computes expectation values by $\mathrm{Tr}(AB) = \int W_A\, W_B\, d\mu$, where $W_A$, $W_B$ are Wigner functions of operators $A$, $B$ and $\mu$ is the normalized Liouville measure on phase space [23]. However, for noncommutative geometry one needs an integral that suppresses infinitesimals of order higher than 1, but the ordinary trace diverges for positive first-order infinitesimals, since

$$\mathrm{Tr}\,|T| = \sum_{k=0}^{\infty} \mu_k(T) = \lim_{n\to\infty} \sigma_n(T) = \infty \quad \text{if } \sigma_n(T) = O(\log n).$$

Dixmier [66] found other (non-normal) tracial functionals on compact operators that are more suitable for our purposes: they are finite on first order infinitesimals and vanish on those of higher order. To find them, we must look more closely at the fine structure of infinitesimal operators; our treatment follows [52, Appendix A]: see also [104, Sec. 7.5].

The algebra $\mathcal{K}$ of compact operators on a separable, infinite-dimensional Hilbert space contains the ideal $\mathcal{L}^1$ of traceclass operators, on which $\|T\|_1 := \text{Tr }|T|$ is a norm, which is larger than the operator norm $\|T\| = \mu_0(T)$. Each partial sum of singular values $\sigma_n$ is a norm on $\mathcal{K}$. In fact,

$$\sigma_n(T) = \sup\{\|TP_n\|_1 : P_n \text{ is a projector of rank } n\}.$$

Notice that $\sigma_n(T) \le n\,\mu_0(T) = n\|T\|$. There is a cute formula [52] that combines these two norms:

$$\sigma_n(T) = \inf\{\|R\|_1 + n\|S\| : R, S \in \mathcal{K};\ R + S = T\}. \tag{5.1}$$

This is worth checking. Clearly, if $T = R + S$, then $\sigma_n(T) \le \sigma_n(R) + \sigma_n(S) \le \|R\|_1 + n\|S\|$. To show that the infimum is attained, we can assume that $T$ is a positive operator, since both sides of (5.1) are unchanged if $R, S, T$ are multiplied on the left by a unitary operator $V$ such that $VT = |T|$. Now let $P_n$ be the projector of rank $n$ whose range is spanned by the eigenvectors of $T$ corresponding to the eigenvalues $\mu_0, \ldots, \mu_{n-1}$. Then $R := (T - \mu_n)P_n$ and $S := \mu_n P_n + T(1 - P_n)$ satisfy $\|R\|_1 = \sum_{k<n}(\mu_k - \mu_n) = \sigma_n(T) - n\mu_n$ and $\|S\| = \mu_n$.

We can think of $\sigma_n(T)$ as the trace of $|T|$ with a cutoff at the scale $n$. This scale need not be an integer; for any scale $\lambda > 0$, we can *define*

$$\sigma_\lambda(T) := \inf\{\|R\|_1 + \lambda\|S\| : R, S \in \mathcal{K};\ R + S = T\}.$$

If $0 < \lambda \le 1$, then $\sigma_\lambda(T) = \lambda\|T\|$. If $\lambda = n + t$ with $0 \le t < 1$, one checks that

$$\sigma_\lambda(T) = (1 - t)\sigma_n(T) + t\sigma_{n+1}(T), \tag{5.2}$$

so $\lambda \mapsto \sigma_\lambda(T)$ is a piecewise linear, increasing, concave function on $(0, \infty)$. Each $\sigma_\lambda$ is a norm by (5.2), and so satisfies the triangle inequality. For *positive* compact operators, there is a triangle inequality in the opposite direction:

$$\sigma_\lambda(A) + \sigma_\mu(B) \le \sigma_{\lambda+\mu}(A + B) \quad \text{if } A, B \ge 0. \tag{5.3}$$

It suffices to check this for integral values $\lambda = m$, $\mu = n$. If $P_m, P_n$ are projectors of respective ranks $m, n$, and if $P = P_m \vee P_n$ is the projector with range $P_m\mathcal{H} + P_n\mathcal{H}$, then

$$\|AP_m\|_1 + \|BP_n\|_1 = \text{Tr}(P_m A P_m + P_n B P_n) \le \text{Tr}(P(A + B)P) = \|(A + B)P\|_1,$$

and (5.3) follows by taking suprema over $m, n$. We get a sandwich of norms:

$$\sigma_\lambda(A + B) \le \sigma_\lambda(A) + \sigma_\lambda(B) \le \sigma_{2\lambda}(A + B) \quad \text{if } A, B \ge 0. \tag{5.4}$$

**The Dixmier ideal.** The *first-order infinitesimals* can now be defined precisely as the following normed ideal of compact operators:

$$\mathcal{L}^{1+} := \left\{ T \in \mathcal{K} : \|T\|_{1+} := \sup_{\lambda \geq e} \frac{\sigma_\lambda(T)}{\log \lambda} < \infty \right\},$$

that obviously includes the traceclass operators $\mathcal{L}^1$. (It is alternatively denoted by $\mathcal{L}^{(1,\infty)}$, a notation coming from the real interpolation theory of Banach spaces [37, IV.B].) On the other hand, if $p > 1$ then $\mathcal{L}^{1+} \subset \mathcal{L}^p$, where the latter is the Schatten ideal of those $T$ such that $\operatorname{Tr} |T|^p < \infty$, for which $\sigma_\lambda(T) = O(\lambda^{1-1/p})$.

Fix $a \in \mathbb{R}$ with $a > e$. If $T \in \mathcal{L}^{1+}$, the function $\lambda \mapsto \sigma_\lambda(T)/\log \lambda$ is *continuous and bounded* on the interval $[a, \infty)$, i.e., it lies in the $C^*$-algebra $C_b[a, \infty)$. We then define the following *Cesàro mean* of this function:

$$\tau_\lambda(T) := \frac{1}{\log \lambda} \int_a^\lambda \frac{\sigma_u(T)}{\log u} \frac{du}{u}.$$

Then $\lambda \mapsto \tau_\lambda(T)$ lies in $C_b[a, \infty)$ also, with upper bound $\|T\|_{1+}$. From (5.4) we can derive that

$$0 \leq \tau_\lambda(A) + \tau_\lambda(B) - \tau_\lambda(A + B) \leq \left( \|A\|_{1+} + \|B\|_{1+} \right) \log 2 \frac{2 + \log \log \lambda}{\log \lambda},$$

so that $\tau_\lambda$ is 'asymptotically additive' on positive elements of $\mathcal{L}^{1+}$.

We get a true additive functional in two more steps. Firstly, let $\dot\tau(A)$ be the class of $\lambda \mapsto \tau_\lambda(A)$ in the quotient $C^*$-algebra $\mathcal{B} := C_b[a, \infty)/C_0[a, \infty)$. Then $\dot\tau$ is an additive, positive-homogeneous map from the positive cone of $\mathcal{L}^{1+}$ into $\mathcal{B}$, and $\dot\tau(UAU^{-1}) = \dot\tau(A)$ for any unitary $U$; therefore it extends to a *linear* map $\dot\tau : \mathcal{L}^{1+} \to \mathcal{B}$ such that $\dot\tau(ST) = \dot\tau(TS)$ for $T \in \mathcal{L}^{1+}$ and any bounded $S$.

Secondly, we follow $\dot\tau$ with any state (i.e., normalized positive linear form) $\omega : \mathcal{B} \to \mathbb{C}$. The composition is a *Dixmier trace*:

$$\operatorname{Tr}_\omega(T) := \omega(\dot\tau(T)).$$

**The noncommutative integral.** The $C^*$-algebra $\mathcal{B}$ is not separable and thus there is no way to *exhibit* any particular state. This problem can be finessed by noticing that a function $f \in C_b[a, \infty)$ has a limit $\lim_{\lambda \to \infty} f(\lambda) = c$ if and only if $\omega(f) = c$ does not depend on $\omega$. Let us say that an operator $T \in \mathcal{L}^{1+}$ is *measurable* if the function $\lambda \mapsto \tau_\lambda(T)$ converges as $\lambda \to \infty$, in which case any $\operatorname{Tr}_\omega(T)$ equals its limit. We denote by $\fint T$ the common value of the Dixmier traces:

$$\fint T := \lim_{\lambda \to \infty} \tau_\lambda(T) \quad \text{if this limit exists.}$$

We call this value the *noncommutative integral* of $T$.

If $T \in \mathcal{K}$ and $\sigma_n(T)/\log n$ converges as $n \to \infty$, then $T$ lies in $\mathcal{L}^{1+}$ and is measurable. Actually, if $T$ is a *positive* operator, then $T$ is measurable *if and only if* $\lim_{n\to\infty} \sigma_n(T)/\log n$ exists [156]. The existence of this limit was shown to be the case for the operators $\not{D}^{-2}$ on the Riemann sphere and $D^{-2}$ on the noncommutative torus, whose integrals we have already computed.

We should do at least one $n$-dimensional integral calculation. For the torus $\mathbb{T}^n := \mathbb{R}^n/(2\pi\mathbb{Z})^n$ with real coordinates $(t^1, \dots, t^n)$, consider its Laplacian

$$\Delta := -\left(\frac{\partial}{\partial t^1}\right)^2 - \cdots - \left(\frac{\partial}{\partial t^n}\right)^2. \qquad (5.5)$$

Its eigenfunctions are $\phi_l(t) := e^{il\cdot t}$ for $l \in \mathbb{Z}^n$. We discard the zero mode $\phi_0$ and regard $\Delta$ as an invertible operator on the orthogonal complement of the constants in $L^2(\mathbb{T}^n)$. The multiplicity $m_\lambda$ of the eigenvalue $\lambda = |l|^2$ is the number of lattice points in $\mathbb{Z}^n$ of length $|l|$. Thus the operator $\Delta^{-s}$ is compact for any $s > 0$; let us compute $\int \Delta^{-s}$.

If $N_r$ is the total number of lattice points with $|l| \leq r$, then $N_{r+dr} - N_r \sim \Omega_n r^{n-1}\,dr$ and $N_r \sim n^{-1}\Omega_n r^n$, where

$$\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \mathrm{vol}(\mathbb{S}^{n-1})$$

is the volume of the unit sphere. For $N = N_R$ we estimate

$$\sigma_{N_R}(\Delta^{-s}) = \sum_{1 \leq |l| \leq R} |l|^{-2s} \sim \int_1^R r^{-2s}\,(N_{r+dr} - N_r) \sim \Omega_n \int_1^R r^{n-2s-1}\,dr.$$

Since $\log N_R \sim n \log R$, we see that $\sigma_N(\Delta^{-s})/\log N$ diverges if $s < n/2$ and converges to 0 if $s > n/2$. For the borderline case $s = n/2$, we get $\sigma_N(\Delta^{-n/2})/\log N \sim \Omega_n/n$, and thus

$$\fint \Delta^{-n/2} = \frac{\Omega_n}{n} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}. \qquad (5.6)$$

The Dirac operator $\not{D} = -i\,\gamma(dx^j)\,\partial/\partial x^j$ on $\mathbb{T}^n$ satisfies $\not{D}^2 = \Delta^S$, a direct sum of $2^{\lfloor n/2 \rfloor}$ copies of the scalar Laplacian $\Delta$, so we may rewrite (5.6) as $\int ds^n = \int |\not{D}|^{-n} = 2^{\lfloor n/2 \rfloor}\Omega_n/n$.

## 5.2 Pseudodifferential operators

Up to now, the computation of $\fint T$ has required a complete determination of the spectrum of $T$. This an onerous task, suitable only for simple examples. An alternative approach is needed, which may allow us to calculate $\fint T$ by a general procedure.

For the commutative case, such a procedure is available: it is the pseudodifferential operator calculus. For the extension of this calculus to the noncommutative case, see for instance [38], [41], [112]. Here we confine our attention to a fairly familiar case: classical pseudodifferential operators ($\Psi$DOs) on compact Riemannian manifolds.

**Definition 10.** A *pseudodifferential operator of order d* over a manifold $M$ is an operator $A\colon L^2(M, E) \to L^2(M, F)$ between two Hilbert spaces of square-integrable sections of Hermitian vector bundles $E, F \to M$, that can be written in local coordinates as

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(x,\xi) u(y)\, d^n y\, d^n\xi, \qquad (5.7)$$

where the *symbol* $a(x,\xi)$ is a matrix of smooth functions whose derivatives satisfy the growth conditions $|\partial_x^\alpha \partial_\xi^\beta a_{ij}(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{d-|\beta|}$. We then say that $A$ is a *classical* $\Psi$DO, written $A \in \Psi^d(M; E, F)$, if its symbol has an asymptotic expansion of the form

$$a(x,\xi) \sim \sum_{j=0}^\infty a_{d-j}(x,\xi) \qquad (5.8)$$

where each $a_r(x,\xi)$ is $r$-*homogeneous* in the variable $\xi$, that is, $a_r(x, t\xi) \equiv t^r\, a_r(x,\xi)$, and the asymptotics means that subtraction of the first $m$ terms lowers the order of growth from $d$ to $d - m$. We refer to [208] for the full story on such operators. Although the terms of the expansion (5.8) are generally coordinate-dependent, the *principal symbol* $a_d(x,\xi)$ is globally defined on the cotangent bundle $T^*M$, except possibly on the zero section $M$. We call the operator $A$ *elliptic* if $a_d(x,\xi)$ is invertible for $\xi \neq 0$.

The spaces $\Psi^d(M; E, F)$ are decreasingly nested; elements of their intersection $\Psi^{-\infty}(M; E, F)$ are called *smoothing* operators. The symbol $a$ determines $A$ up to a smoothing operator. The quotient $\mathcal{P} := \Psi^\infty(M; E, E)/\Psi^{-\infty}(M; E, E)$ is an algebra when $E = F$. The product $AB = C$ of $\Psi$DOs corresponds to the composition of symbols given by the expansion

$$c(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a\, \partial_x^\alpha b.$$

From the leading term, we see that if $A \in \Psi^d(M; E, E)$, $B \in \Psi^r(M; E, E)$, then $AB \in \Psi^{d+r}(M; E, E)$. Also, if $P = [A, B]$ is a commutator in $\Psi^\infty(M; E, E)$, then

$$p(x,\xi) \sim \sum_{|\alpha|>0} \frac{(-i)^{|\alpha|}}{\alpha!} \left( \partial_\xi^\alpha a\, \partial_x^\alpha b - \partial_\xi^\alpha b\, \partial_x^\alpha a \right) \qquad (5.9)$$

in the scalar case where $E$ is a line bundle, since then the term of order $d + r$ cancels.

## 5.3 The Wodzicki residue

If $M$ is an $n$-dimensional manifold, the term of order $(-n)$ of the expansion (5.8) has a special significance. It is coordinate-dependent, so let us fix a coordinate domain $U \subset M$ over which the cotangent bundle is trivial, and consider the matrix trace $\operatorname{tr} a_{-n}(x, \xi)$ as a smooth function on $T^*U \setminus U$, omitting the zero section. Then

$$\alpha := \operatorname{tr} a_{-n}(x, \xi) \, d\xi_1 \wedge \cdots \wedge d\xi_n \wedge dx^1 \wedge \cdots \wedge dx^n$$

is invariant under the dilations $\xi \mapsto t\xi$, because $\operatorname{tr} a_{-n}(x, \xi)$ is homogeneous of degree $(-n)$. Thus, if $R = \sum_j \xi_j \, \partial/\partial \xi_j$ is the radial vector field on $T^*U \setminus U$ that generates these dilations, then $d \, \iota_R \alpha = \mathcal{L}_R \, \alpha = 0$, so $\iota_R \alpha$ is a closed $(2n-1)$-form on $T^*U \setminus U$. Here $\mathcal{L}_R = \iota_R d + d \, \iota_R$ is the Lie derivative. On abbreviating

$$d^n x := dx^1 \wedge \cdots \wedge dx^n, \quad \sigma_\xi := \sum_{j=1}^{n} (-1)^{j-1} \xi_j \, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n,$$

we find that $\iota_R \alpha = \operatorname{tr} a_{-n}(x, \xi) \, \sigma_\xi \wedge d^n x$.

Now, $\sigma_\xi$ restricts to the volume form on the unit sphere $|\xi| = 1$ in each $T_x^* M$. On integrating $\iota_R \alpha$ over these spheres, we get a quantity that transforms under coordinate changes $x \mapsto y = \phi(x)$, $\xi \mapsto \eta = \phi'(x)^t \xi$, $a_{-n}(x, \xi) \mapsto \tilde{a}_{-n}(y, \eta)$ as follows [84], [104]:

$$\int_{|\eta|=1} \operatorname{tr} \tilde{a}_{-n}(y, \eta) \, \sigma_\eta = |\det \phi'(x)| \int_{|\xi|=1} \operatorname{tr} a_{-n}(x, \xi) \, \sigma_\xi. \tag{5.10}$$

The absolute value of the Jacobian $\det \phi'(x)$ appears here because if $\phi'(x)^t$ reverses the orientation on the unit sphere in $T_x^* M$ then the integral over the sphere also changes sign.

**The Wresidue density.** As a consequence of (5.10), we get a 1-*density* on $M$, denoted $\operatorname{wres} A$, whose local expression on any coordinate chart is

$$\operatorname{wres}_x A := \left( \int_{|\xi|=1} \operatorname{tr} a_{-n}(x, \xi) \, \sigma_\xi \right) dx^1 \wedge \cdots \wedge dx^n. \tag{5.11}$$

This is the *Wodzicki residue density*. By integrating it over $M$, we get the *Wodzicki residue* [84], [124], [223]:

$$\operatorname{Wres} A := \int_M \operatorname{wres} A = \int_{\mathbb{S}^* M} \iota_R \alpha = \int_{\mathbb{S}^* M} \operatorname{tr} a_{-n}(x, \xi) \, \sigma_\xi \, d^n x. \tag{5.12}$$

Here $\mathbb{S}^* M := \{ (x, \xi) \in T^* M : g_x^{-1}(\xi, \xi) = 1 \}$ is the 'cosphere bundle' over $M$. The integral (5.12) may diverge for some $A$; we shall shortly identify its domain.

**The tracial property of the Wresidue.** It turns out that Wres is a *trace* on the algebra $\mathcal{P}$ of classical symbols, i.e., that $\text{Wres}[A, B] = 0$ always, provided that $\dim M > 1$. The reason is that each term in the expansion is a finite sum of derivatives $\partial p / \partial x^j + \partial q / \partial \xi_j$. For instance, in the scalar case the leading term of (5.9) is

$$-i \left( \frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x^j} - \frac{\partial b}{\partial \xi_j} \frac{\partial a}{\partial x^j} \right) = \frac{\partial}{\partial x^j} \left( i a \frac{\partial b}{\partial \xi_j} \right) - \frac{\partial}{\partial \xi_j} \left( i a \frac{\partial b}{\partial x^j} \right).$$

We can assume that $a$, $b$ are supported on a compact subset of a chart domain $U$ of $M$ (since we can later patch together with a partition of unity), so that all $(-n)$-homogeneous terms of type $\partial p / \partial x^j$ have zero integral over $\mathbb{S}^* M$. Since we are integrating a closed $(2n - 1)$-form over $\mathbb{S}^{n-1} \times U$, we get the same result by integration over the cylinder $\mathbb{S}^{n-2} \times \mathbb{R} \times U$: these are cohomologous cycles in $(\mathbb{R}^n \setminus \{0\}) \times U$. For any term of the form $\partial q / \partial \xi_j$, where $q$ is $(-n + 1)$-homogeneous, we then get

$$\int_{|\xi|=1} \frac{\partial q}{\partial \xi_j} = \pm \int_{|\xi'|=1} \int_{-\infty}^{\infty} \frac{\partial q}{\partial \xi_j} \, d\xi_j \, \sigma_{\xi'} = 0$$

if $\xi' := (\xi_1, \ldots, \widehat{\xi_j}, \ldots, \xi_n)$, since $q(x, \xi) \to 0$ as $\xi_j \to \pm\infty$ because $-n + 1 < 0$.

The crucial property of Wres is that, up to scalar multiples, it is the *unique* trace on the algebra $\mathcal{P}$. We give the gist of the elementary proof of this by Fedosov *et al.* [84]. From the symbol calculus, derivatives are commutators, since

$$[x^j, a] = i \frac{\partial a}{\partial \xi_j}, \quad [\xi_j, a] = -i \frac{\partial a}{\partial x^j}$$

in view of (5.9). Hence any trace $T$ on symbols must vanish on derivatives. For $r \neq -n$, each $r$-homogeneous term $a_r(x, \xi)$ is a derivative, since $\partial / \partial \xi_j (\xi_j a_r) = (n + r) a$ by Euler's theorem. Furthermore, one can show that after averaging over spheres, $\tilde{a}_{-n}(x) := \Omega_n^{-1} \int_{|\xi|=1} a_{-n}(x, \xi) \, \sigma_\xi$, the centred $(-n)$-homogeneous term

$$\text{tr} \, a_{-n}(x, \xi) - \text{tr} \, \tilde{a}_{-n}(x) \, |\xi|^{-n}$$

is a finite sum of derivatives. The upshot is that $T(a) = T(\tilde{a}_{-n}(x) \, |\xi|^{-n})$ is a linear functional of $\text{tr} \, \tilde{a}_{-n}(x)$ that kills derivatives, so it must be of the form

$$T(a) = C \int_U \text{tr} \, \tilde{a}_{-n}(x) \, d^n x = C \, \text{Wres} \, A.$$

## 5.4 The trace theorem

This uniqueness of the trace was exploited by Connes [35], who saw how the Dixmier traces fit into this picture. The point is that pseudodifferential operators of low enough

order over a compact manifold are already compact operators [208], so that any Dixmier trace $\mathrm{Tr}_\omega$ defines a trace on $\Psi^{-n}(M)$; and they all define the same trace, since they coincide on measurable operators. Thus all $\Psi$DOs of order $(-n)$ are measurable, and the noncommutative integral is a multiple of Wres. It remains only to compute the proportionality constant. Moreover, since we can reduce to local calculations by patching with partitions of unity, this constant must be the same for all manifolds of a given dimension.

To find it, we can use the power $\Delta^{-n/2}$ of the Laplacian on the torus $\mathbb{T}^n$, with its translation-invariant flat metric, whose noncommutative integral we already know (5.6). Now $\Delta$ is a differential operator (5.5), with symbol $b(x,\xi) = g^{-1}(\xi,\xi) = |\xi|^2$. Thus $\Delta^{-n/2}$ is a $\Psi$DO with symbol $|\xi|^{-n}$, which is $(-n)$-homogeneous; and $|\xi|^{-n} \equiv 1$ on the cosphere bundle $\mathbb{S}^*M$. Thus

$$\mathrm{Wres}\,\Delta^{-n/2} = \int_{\mathbb{T}^n} \int_{|\xi|=1} \sigma_\xi\, d^n x = (2\pi)^n\, \Omega_n.$$

On comparing (5.6), we see that the proportionality constant is $1/n(2\pi)^n$, thus

$$\fint A = \frac{1}{n\,(2\pi)^n}\, \mathrm{Wres}\, A \tag{5.13}$$

for any $\Psi$DO of order $(-n)$ or lower. This is Connes' trace theorem [35].

**The commutative integral.** Suppose that $M$ is an $n$-dimensional spin manifold, with Dirac operator $\slashed{D}$. The operator $|\slashed{D}|^{-n}$ is a first-order infinitesimal and is also a pseudodifferential operator of order $(-n)$. Indeed, $\slashed{D}$ acts on spinors with principal symbol $\gamma(\xi)$, so $\slashed{D}^2$ has principal symbol $g^{-1}(\xi,\xi)\,1_N$, where $g$ is the Riemannian metric; this is a scalar matrix whose size $N = 2^{\lfloor n/2 \rfloor}$ is the rank of the spinor bundle. The principal symbol of $|\slashed{D}|^{-n} = (\slashed{D}^2)^{-n/2}$ is then given locally by $\sqrt{\det g(x)}\,|\xi|^{-n}\,1_N$.

More generally, when $a \in C^\infty(M)$ is represented as a multiplication operator on spinors, the operator $A = a\,|\slashed{D}|^{-n}$ is also pseudodifferential of order $(-n)$, with principal symbol $a_{-n}(x,\xi) := a(x)\sqrt{\det g(x)}\,|\xi|^{-n}\,1_N$. Let us be mindful that the Riemannian volume form is $\Omega = \sqrt{\det g(x)}\,dx^1 \wedge \cdots \wedge dx^n$. Invoking (5.13) and (5.12), we end up with

$$\fint a\,|\slashed{D}|^{-n} = \frac{1}{n\,(2\pi)^n}\, \mathrm{Wres}\, a\,|\slashed{D}|^{-n} = \frac{1}{n\,(2\pi)^n} \int_{\mathbb{S}^*M} \mathrm{tr}\, a_{-n}(x,\xi)\,\sigma_\xi\, d^n x$$
$$= \frac{2^{\lfloor n/2 \rfloor}\,\Omega_n}{n\,(2\pi)^n} \int_M a(x)\,\Omega. \tag{5.14}$$

Thus the *ordinary integral* on functions, determined by the orientation class $[\Omega]$, is

$$\int_M a\,\Omega = \begin{cases} m!\,(2\pi)^m \fint a\,\slashed{D}^{-2m} & \text{if } \dim M = 2m \text{ is even,} \\ (2m+1)!!\,\pi^{m+1} \fint a\,|\slashed{D}|^{-2m-1} & \text{if } \dim M = 2m+1 \text{ is odd.} \end{cases}$$

In particular, since orientable 2-dimensional manifolds always admit a spin structure [96], the area of a surface ($m = 1$) is $2\pi \!\!\!\!\fint \!\! D\!\!\!\!/^{-2}$, as we had previously asserted.

## 5.5 Integrals and zeta residues

We have not yet explained why the Wodzicki functional is called a residue. It was originally discovered as the Cauchy residue of a zeta function: see Wodzicki's introductory remarks in [223]. Indeed, the following formula can be established [215] with the help of Seeley's symbol calculus for complex powers of a positive elliptic $\Psi$DO [196]:

$$\operatorname*{Res}_{s=1} \zeta_A(s) = \frac{1}{n\,(2\pi)^n}\, \text{Wres}\, A,$$

where the zeta function of a positive compact operator $A$ with eigenvalues $\lambda_k(A)$ may be defined as

$$\zeta_A(s) := \sum_{k=1}^{\infty} \lambda_k(A)^s \quad \text{for } \Re s > \text{some } s_0$$

and extended to a meromorphic function on $\mathbb{C}$ by analytic continuation.

In view of (5.13), it should be possible to prove directly that this zeta residue coincides with the noncommutative integral of measurable infinitesimals:

$$\operatorname*{Res}_{s=1} \zeta_A(s) = \fint A. \tag{5.15}$$

This can indeed be achieved by known Tauberian theorems; see, for instance, [37, IV.2.$\beta$, Prop. 4], [215, Sec. 2] or [104, Sec. 7.6]. Rather than repeat these technical proofs here, we give instead a heuristic argument based on the delta-function calculus of [81], that shows why (5.15) is to be expected.

Let us take $A$ to be a compact positive operator whose eigenvalues satisfy $\lambda_k(A) \sim L/k$ as $k \to \infty$. Then $\sigma_n(A) \sim L \log n$ as $n \to \infty$, so that $A$ is measurable with $\fint A = L$. In the particular case of the operator $R$ for which $\lambda_k(R) = 1/k$ for all $k$, $\zeta_R(s)$ is precisely the Riemann zeta function.

On the other hand, let us examine an interesting distribution on $\mathbb{R}$, the 'Dirac comb' $\sum_{k\in\mathbb{Z}} \delta(x - k)$. It is periodic and its mean value is 1; therefore $f(x) := \sum_{k\in\mathbb{Z}} \delta(x - k) - 1$ is a periodic distribution of mean zero. It can then be shown [80] that the *moments* $\mu_m := \int_{-\infty}^{\infty} f(x)\, x^m\, dx$ exist for all $m$. The same is true if we cut off the $x$-axis at $x = 1$ with the Heaviside function $\theta(x - 1)$: the distribution

$$f_R(x) := \sum_{k=1}^{\infty} \delta(x - k) - \theta(x - 1)$$

has moments of all orders, and the function

$$Z_R(s) := \int_{\mathbb{R}} f_R(x)\, x^{-s}\, dx = \sum_{k=1}^{\infty} \frac{1}{k^s} - \int_1^{\infty} x^{-s}\, dx = \zeta_R(s) - \frac{1}{s-1}$$

is an entire analytic function of $s$. (Notice how this argument shows that $\zeta_R$ is mero-morphic with a single simple pole at $s = 1$, whose residue is 1.)

Now since $L/\lambda_k(A) \sim k$ as $k \to \infty$, we replace $\delta(x - k)$ by $\delta(x - L/\lambda_k)$ and define

$$f_A(x) := \sum_{k=1}^{\infty} \delta(x - L/\lambda_k) - \theta(x - 1).$$

This suggests that

$$Z_A(s) := \int_{\mathbb{R}} f_A(x)\, x^{-s}\, dx = L^{-s} \sum_{k=1}^{\infty} \lambda_k^s - \int_1^{\infty} x^{-s}\, dx = L^{-s} \zeta_A(s) - \frac{1}{s-1}.$$

From this we conclude that $\zeta_A(s) = L^s Z_A(s) + L^s/(s-1)$ is meromorphic, analytic for $\Re s > 1$, and has a simple pole at $s = 1$ with residue

$$\operatorname*{Res}_{s=1} \zeta_A(s) = L = \fint A.$$

This surprising nexus between the Wodzicki functional, the noncommutative inte-gral and the zeta-function residue suggests that the first two may be profitably used in quantum field theory; see [72] for an example of that.

# 6

# Quantization and the tangent groupoid

We now turn aside from the study of spin geometries to explore an issue of a more topological nature, namely, the extent to which noncommutative methods allow us to achieve contact with the quantum world. To begin with, the facile but oft-repeated phrase, 'noncommutative = quantum', must be disregarded. As the story of the Connes–Lott model shows, a perfectly noncommutative geometry may be employed to produce Lagrangians for physical models at the classical level only [37, VI.5]; quantization must then proceed from this starting point. Nevertheless, the foundations of quantum mechanics do throw up noncommutative geometries, as we have seen with the torus. Also, the integrality features of the pairing of cyclic cohomology with $K$-theory give genuine examples of quantizing. So, it is worthwhile to ask: what is the relation of known quantization procedures with noncommutative geometry?

The first step in quantizing a system with finitely many degrees of freedom is to place the classical and quantum descriptions of the system on the same footing; the second step is to draw an unbroken line between these descriptions. In conventional quantum mechanics, the simplest such method is the Wigner–Weyl or Moyal quantization, which consists in 'deforming' an algebra of *functions on phase space* to an algebra of *operator kernels*. In noncommutative geometry, there is a device that accomplishes this in a most economical manner, namely the *tangent groupoid* of a configuration space.

## 6.1 Moyal quantizers and the Moyal deformation

Since [8] and [11] at least, deformations of algebras have been related to the physics of quantization: see [146] for the full panorama. Here we first sketch the general scheme [99], and then illustrate it with the simplest possible example, namely, the Moyal quantization in terms of the Schrödinger representation of ordinary quantum mechanics for spinless, nonrelativistic particles.

**Definition 11.** Let $X$ be a smooth symplectic manifold, $\mu$ (an appropriate multiple of) the associated Liouville measure, and $\mathcal{H}$ a Hilbert space. A *Moyal quantizer* for the triple $(X, \mu, \mathcal{H})$ is a map $\Omega$ of the phase space $X$ into the space of selfadjoint operators on $\mathcal{H}$ satisfying

$$\operatorname{Tr} \Omega(u) = 1, \tag{6.1a}$$

$$\operatorname{Tr}\big[\Omega(u)\Omega(v)\big] = \delta(u - v), \tag{6.1b}$$

for $u, v \in X$, at least in a distributional sense; and such that $\{\Omega(u) : u \in X\}$ spans a weakly dense subspace of $\mathcal{L}(\mathcal{H})$. More precisely, the notation '$\delta(u - v)$' means the (distributional) reproducing kernel for the measure $\mu$. An essentially equivalent definition, in the equivariant context, was introduced first in [214].

For the proud owner of a Moyal quantizer, all quantization problems are solved in principle. *Quantization* of a sufficiently regular function $a$ on $X$ is effected by

$$a \mapsto Q(a) := \int_X a(u)\,\Omega(u)\,\mu(u), \tag{6.2}$$

and *dequantization* of any operator $A \in \mathcal{L}(\mathcal{H})$ by

$$A \mapsto W_A, \quad \text{with } W_A(u) := \text{Tr}\big[A\,\Omega(u)\big].$$

This makes $W_I = 1$ automatically. Moreover, $W_{Q(a)} = a$ by (6.1b) and therefore $Q(W_A) = A$ by irreducibility, from which it follows that $Q(1) = I$, i.e., $\int_X \Omega(u)\,\mu(u) = I$.

We can reformulate (6.1) as

$$\text{Tr}\,Q(a) = \int_X a(u)\,\mu(u), \tag{6.3a}$$

$$\text{Tr}\big[Q(a)Q(b)\big] = \int_X a(u)b(u)\,\mu(u), \tag{6.3b}$$

for *real* functions $a, b \in L^2(X, \mu)$.

Moyal quantizers are essentially unique and understandably difficult to find [99]. The standard example is given by the phase space $T^*(\mathbb{R}^n)$ with the canonical symplectic structure. With Planck constant $\hbar > 0$ and coordinates $u = (q, \xi)$, we take $\mu(q, \xi) := (2\pi\hbar)^{-n}\,d^n q\,d^n \xi$. We also take $\mathcal{H} = L^2(\mathbb{R}^n)$. Then the Moyal quantizer on $T^*(\mathbb{R}^n)$ is given by a family of symmetries $\Omega^\hbar(q, \xi)$; explicitly, in the Schrödinger representation:

$$\big[\Omega^\hbar(q, \xi)f\big](x) = 2^n\,e^{2i\xi(x-q)/\hbar}\,f(2q - x). \tag{6.4}$$

The *Moyal product* $a \star_\hbar b$ of two elements $a, b$ of $\mathcal{S}(T^*\mathbb{R}^n)$, say, is defined by the requirement that $Q(a \star_\hbar b) = Q(a)Q(b)$. In this case,

$$(a \star_\hbar b)(u) = \iint L^\hbar(u, v, w)\,a(v)b(w)\,\mu(v)\,\mu(w), \tag{6.5}$$

with

$$
\begin{aligned}
L^\hbar(u, v, w) &:= \text{Tr}\big[\Omega^\hbar(u)\,\Omega^\hbar(v)\,\Omega^\hbar(w)\big] \\
&= 2^{2n} \exp\Big\{-\frac{2i}{\hbar}\big(s(u, v) + s(v, w) + s(w, u)\big)\Big\},
\end{aligned}
$$

where $s(u, u') := q\xi' - q'\xi$ is the linear symplectic form on $T^*\mathbb{R}^n$. Note that $\int a \star_\hbar b = \int ab$. By duality, then, the quantization rule can be extended to very large spaces of functions and distributions [91], [102], [183], [213].

Moyal quantization has other interesting uniqueness properties; for instance, it can be also uniquely obtained by demanding equivariance with respect to the linear symplectic group (on introducing the metaplectic representation) [202].

**Asymptotic morphisms.** $K$-theory, being homotopy invariant, is fairly rigid under algebra deformations. $K_0$ is a functor; for each $*$-homomorphism of $C^*$-algebras $\phi : A \to B$ there is a group homomorphism $\phi_* : K_0(A) \to K_0(B)$. However, there is no need to ask for $*$-homomorphisms; is order to obtain $K$-theory maps, it is enough to construct *asymptotic* morphisms of the $C^*$-algebras. These were introduced by Connes and Higson in [48].

**Definition 12.** Let $A$, $B$ be $C^*$-algebras and $\hbar_0$ a positive real number. An *asymptotic morphism* from $A$ to $B$ is a family of maps $T = \{T_\hbar : A \to B : 0 < \hbar \leq \hbar_0\}$, such that $\hbar \mapsto T_\hbar(a)$ is norm-continuous on $(0, \hbar_0]$ for each $a \in A$, and such that, for $a, b \in A$ and $\lambda \in \mathbb{C}$, the following norm limits apply:

$$\lim_{\hbar \downarrow 0}\big(T_\hbar(a) + \lambda T_\hbar(b) - T_\hbar(a + \lambda b)\big) = 0,$$

$$\lim_{\hbar \downarrow 0}\big(T_\hbar(a)^* - T_\hbar(a^*)\big) = 0,$$

$$\lim_{\hbar \downarrow 0}\big(T_\hbar(ab) - T_\hbar(a)T_\hbar(b)\big) = 0.$$

Two asymptotic morphisms $T$, $S$ are equivalent if $\lim_{\hbar \downarrow 0}(T_\hbar(a) - S_\hbar(a)) = 0$ for all $a \in A$. Therefore, by setting $\widetilde{T}(a)_\hbar := T_\hbar(a)$, equivalence classes of asymptotic morphisms from $A$ to $B$ determine $*$-homomorphisms from $A$ to the quotient $C^*$-algebra $\widetilde{B} := C_b((0, \hbar_0], B)/C_0((0, \hbar_0], B)$. We remark that our $T_\hbar$ is the $\phi_{1/\hbar}$ of [37], [48].

In examples, it is enough that the maps $T_\hbar$ be defined on a pre-$C^*$-algebra $\mathcal{A}$ that is dense in $A$, since the $*$-homomorphism $\widetilde{T} : \mathcal{A} \to \widetilde{B}$ automatically extends to $A$ [104, Lemma 3.41]. Recovery of an asymptotic morphism from the extended $\widetilde{T} : A \to \widetilde{B}$ requires some delicate use of selection theorems, but is always possible: see [48] or [104, Sec. 3.4].

In order to define maps of $K$-theory groups, it is enough to have asymptotic morphisms. This works as follows [111]: first extend $T_\hbar$ entrywise to an asymptotic morphism from $M_m(A)$ to $M_m(B)$. If $p$ is a projector in $M_m(A)$, then by functional calculus there is a continuous family of projectors $\{q_\hbar : 0 < \hbar \leq \hbar_0\}$, whose $K$-theory class is well defined, such that $\|T_\hbar(p) - q_\hbar\| \to 0$ as $\hbar \downarrow 0$. Define $T_* : K_0(A) \to K_0(B)$ by $T_*[p] := [q_{\hbar_0}]$.

If $A$ and $B$ are two $C^*$-algebras, a *strong deformation* from $A$ to $B$ is a *continuous field of $C^*$-algebras* $\{A_\hbar : 0 \leq \hbar \leq \hbar_0\}$ in the sense of [67], such that $A_0 = A$ and

$A_\hbar = B$ for $\hbar > 0$. The definition of a continuous field involves specifying the space $\Gamma$ of norm-continuous sections $\hbar \mapsto s(\hbar) \in A_\hbar$, and guarantees that for any $a \in A_0$ there is such a section $s_a$ with $s_a(0) = a$. Such a deformation gives rise to an asymptotic morphism from $A$ to $B$ by setting $T_\hbar(a) := s_a(\hbar)$.

**The Moyal asymptotic morphism.** This is the family of linear maps from $C_0(T^*\mathbb{R}^n)$ to $\mathcal{K}(L^2(\mathbb{R}^n))$, given in terms of integral kernels by:

$$\big[T_\hbar(a)f\big](x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} a\Big(\frac{x+y}{2}, \xi\Big) e^{i\xi(x-y)/\hbar} f(y)\, d^n y\, d^n\xi, \qquad (6.6)$$

on substituting the quantizer (6.4) into (6.2). Here $a$ is an element of the pre-$C^*$-algebra $\mathcal{S}(T^*\mathbb{R}^n)$, to begin with, so the integral is well defined and the operators $T_\hbar(a)$ are in fact trace-class; they are uniformly bounded in $\hbar$. The extension to $C_0(T^*\mathbb{R}^n)$ follows, but need not be described directly by (6.6); in particular, its image goes beyond the trace-class operators. It is clear that $T_\hbar(a)$ is the adjoint of $T_\hbar(a^*)$. This formula is often called the *Moyal deformation*, or the 'Heisenberg deformation' [37, II.B.$\varepsilon$].

In order to check the continuity of the deformation at $\hbar = 0$, one can use, for instance, the following distributional identity:

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-n} e^{ixy/\varepsilon} = (2\pi)^n\, \delta(x)\, \delta(y), \quad \text{for } x, y \in \mathbb{R}^n,$$

from the theory of Fresnel integrals. Thus, $\lim_{\hbar\downarrow 0} L^\hbar(u, v, w) = \delta(u - v)\, \delta(u - w)$, so the Moyal product reduces to the ordinary product in the limit $\hbar \downarrow 0$.

## 6.2 Smooth groupoids

By definition, a *groupoid* $G \rightrightarrows U$ is a small category in which every morphism has an inverse. Its set of objects is $U$ and its set of morphisms is $G$. In practice, this means that we are given a set $G$, another set $U$ of 'units' with an inclusion $i \colon U \hookrightarrow G$, two 'range and source' maps $r, s \colon G \to U$, and a composition law $G^{(2)} \to G$ with domain

$$G^{(2)} := \{(g, h) : s(g) = r(h)\} \subseteq G \times G,$$

subject to the following rules:

(i)  $r(gh) = r(g)$ and $s(gh) = s(h)$ if $(g, h) \in G^{(2)}$;

(ii)  if $u \in U$ then $r(u) = s(u) = u$;

(iii)  $r(g)g = g = gs(g)$ for all $g \in G$;

(iv)  $(gh)k = g(hk)$ if $(g, h) \in G^{(2)}$ and $(h, k) \in G^{(2)}$;

(v) each $g \in G$ has an 'inverse' $g^{-1}$, satisfying $gg^{-1} = r(g)$ and $g^{-1}g = s(g)$.

Any group $G$ is a groupoid, with $U = \{1\}$. On the other hand, any set $X$ is a groupoid, with $G = U = X$ and trivial composition law $x \cdot x = x$. Less trivial examples include graphs of equivalence relations, group actions, and vector bundles with fibrewise addition. An important example is the *pair groupoid* of a set $X$. Take $G = X \times X$ and $U = X$, included in $X \times X$ as the diagonal subset $\Delta_X := \{(x, x) : x \in X\}$. Define $r(x, y) := x$, $s(x, y) := y$. Then $(x, y)^{-1} = (y, x)$ and the composition law is

$$(x, y) \cdot (y, z) = (x, z).$$

Notice that a disjoint union of groupoids is itself a groupoid.

**Definition 13.** A *smooth groupoid* is a groupoid $G \rightrightarrows U$ where $G$, $U$ and $G^{(2)}$ are manifolds (possibly with boundaries), such that the inclusion $i : U \hookrightarrow G$ and the composition and inversion operations are smooth maps, and the maps $r, s \colon G \to U$ are submersions.

Thus, the tangent maps $T_g r$ and $T_g s$ are surjective at each $g \in G$. In particular, this implies that $\operatorname{rank} r = \operatorname{rank} s = \dim U$. Relevant examples are a Lie group, a vector bundle over a smooth manifold, and the pair groupoid $M \times M$ of a smooth manifold.

One can add more structure, if desired. For instance, there are symplectic groupoids, where $G$ is a symplectic manifold and $U$ is a Lagrangian submanifold. These can be used to connect the Kostant–Kirillov–Souriau theory of geometric quantization with Moyal quantization [103], [222].

**Convolution on groupoids.** Functions on groupoids can be convolved in the following way [127], [168], [176]. Suppose that on each $r$-fibre $G^u := \{g \in G : r(g) = u\}$ there is given a measure $\lambda^u$ so that $\lambda^{r(g)} = g\lambda^{s(g)}$ for all $g \in G$; such a family of measures is called a 'Haar system' for $G$. The inversion map $g \mapsto g^{-1}$ carries each $\lambda^u$ to a measure $\lambda_u$ on the $s$-fibre $G_u := \{g \in G : s(g) = u\}$. The *convolution* of two functions $a$, $b$ on $G$ is then defined by

$$(a * b)(g) := \int_{hk=g} a(h)\, b(k) := \int_{G^{r(g)}} a(h)\, b(h^{-1}g)\, d\lambda^{r(g)}(h).$$

**Definition 14.** The *reduced C\*-algebra* of the smooth groupoid $G \rightrightarrows U$ with a given Haar system is the algebra $C_r^*(G)$ obtained by completing $C_c^\infty(G)$ in the norm $\|a\| := \sup_{u \in U} \|\pi_u(a)\|$, where $\pi_u$ is the representation of $C_c^\infty(G)$ on the Hilbert space $L^2(G_u, \lambda_u)$ given by

$$\left[\pi_u(a)\xi\right](g) := \int_{hk=g} a(h)\, \xi(k) := \int_{G^{r(g)}} a(h)\xi(h^{-1}g)\, d\lambda^{r(g)}(h) \quad \text{for } g \in G_u.$$

There is a more canonical procedure to define convolution, if one wishes to avoid hunting for suitable measures $\lambda^u$, which is to take $a$, $b$ to be not functions but *half-densities* on $G$ [37, II.5]. These form a complex line bundle $\Omega^{1/2} \to G$ and one replaces $C_c^\infty(G)$ by the compactly supported smooth sections $C_c^\infty(G, \Omega^{1/2})$ in defining $C_r^*(G)$. However, for the examples considered here, the previous definition will do.

On the pair groupoid $M \times M$ of an oriented Riemannian manifold $M$, we obtain just the convolution of integral kernels:

$$(a * b)(x, z) := \int_M a(x, y) \, b(y, z) \, \Omega(y),$$

where $\Omega(y) = \sqrt{\det g(y)} \, d^n y$ is the volume form on $M$. Here $C_c^\infty(M \times M)$ is the algebra of kernels of smoothing operators on $L^2(M)$, and the $C^*$-algebra $C_r^*(M \times M)$ is its completion, acting as integral kernels on $L^2(M)$, so that $C_r^*(M \times M) \simeq \mathcal{K}$.

The tangent bundle $G = TM$ is a groupoid, whose operation is addition of tangent vectors at the same point, and $U = M$ is included in $TM$ as the zero section; the maps $r$ and $s$ both coincide with the bundle projection $\tau \colon TM \to M$. In this case, $C_r^*(TM)$ is the completion of the convolution algebra

$$(a * b)(q, v) := \int_{T_q M} a(q, u) \, b(q, v - u) \, \sqrt{\det g(q)} \, d^n u,$$

where we may take $a(q, \cdot)$ and $b(q, \cdot)$ in $C_c^\infty(T_q M)$. The *Fourier transform*

$$\mathcal{F} a(q, \xi) := \int_{T_q M} e^{-i\xi v} a(q, v) \, \sqrt{\det g(q)} \, d^n v$$

replaces convolution by the ordinary product on the total space $T^*M$ of the cotangent bundle. This gives an isomorphism from $C_r^*(TM)$ to $C_0(T^*M)$, also called $\mathcal{F}$, with inverse:

$$\mathcal{F}^{-1} b(q, v) = (2\pi)^{-n} \int_{T_q^* M} e^{i\xi v} b(q, \xi) \det^{-1/2}(g(q)) \, d^n \xi.$$

We write $\mu_q(v) := \det^{1/2}(g(q)) \, d^n v$ and $\mu_q(\xi) := (2\pi)^{-n} \det^{-1/2}(g(q)) \, d^n \xi$.

## 6.3 The tangent groupoid

The asymptotic morphism involved in Moyal quantization can be given a concrete geometrical realization and a far-reaching generalization, by the concept of a tangent groupoid.

**Definition 15.** To build the tangent groupoid of a manifold $M$, we first form the disjoint union $G' := M \times M \times (0, 1]$ of copies of the pair groupoid of $M$ parametrized by $0 < \hbar \leq 1$. Its unit set is $U' = M \times (0, 1]$. We also take $G'' := TM$, whose unit set is $U'' = M$. The *tangent groupoid* of $M$ is the disjoint union $G_M := G' \uplus G''$, with $U_M := U' \uplus U''$ as unit set, and composition law given by

$$(x, y, \hbar) \cdot (y, z, \hbar) := (x, z, \hbar) \qquad \text{for } \hbar > 0 \text{ and } x, y, z \in M,$$
$$(q, v_q) \cdot (q, w_q) := (q, v_q + w_q) \quad \text{for } q \in M \text{ and } v_q, w_q \in T_z M.$$

Also, $(x, y, \hbar)^{-1} := (y, x, \hbar)$ and $(q, v_q)^{-1} := (q, -v_q)$.

The tangent groupoid can be given a structure of smooth groupoid in such a way that $G_M$ is a manifold with boundary, $G'$ contains the interior of the manifold and $G''$ is the nontrivial boundary.

In order to see that, let us first recall what is meant by the *normal bundle* over a submanifold $R$ of a manifold $M$ [64]. If $j: R \to M$ is the inclusion map, the tangent bundle of $M$ restricts to $R$ as the pullback $j^*(TM) = TM\big|_R$; this is a vector bundle over $R$, of which $TR$ is a subbundle, and the normal bundle is the quotient $N^j := j^*(TM)/TR$. When $M$ has a Riemannian metric, we may identify the fibre $N_q^j$ to the orthogonal complement of $T_q R$ in $T_q M$ and thus regard $N^j$ as a subbundle of $TM\big|_R$.

At each $q \in R$, the exponential map $\exp_q$ is one-to-one from a small enough ball in $N_q^j$ into $M$. If the submanifold $R$ is compact, then for some $\varepsilon > 0$, the map $(q, v_q) \mapsto \exp_q(v_q)$ with $|v_q| < \varepsilon$ is a diffeomorphism from a neighbourhood of the zero section in $N^j$ to a neighbourhood of $R$ in $M$ (this is the tubular neighbourhood theorem).

Now consider the normal bundle $N^\Delta$ associated to the diagonal embedding $\Delta: M \to M \times M$. We can identify $\Delta^* T(M \times M)$ to $TM \oplus TM$, and thereby the normal bundle over $M$ is identified with

$$N^\Delta = \left\{ \left( \Delta(q), \tfrac{1}{2} v_q, -\tfrac{1}{2} v_q \right) : (q, v_q) \in TM \right\},$$

which gives an obvious isomorphism between $TM$ and $N^\Delta$.

As in the tubular neighbourhood theorem, we can (if $M$ is compact, at any rate) find a diffeomorphism $\phi: V_1 \to V_2$ between an open neighbourhood $V_1$ of $M$ in $N^\Delta$ (considering $M \subset N^\Delta$ as the zero section) and an open neighborhood $V_2$ of $\Delta(M)$ in $M \times M$. Explicitly, we can find $r_0 > 0$ so that

$$\phi(\Delta(q), v_q, -v_q) := \left( \exp_q\left( \tfrac{1}{2} v_q \right), \exp_q\left( -\tfrac{1}{2} v_q \right) \right)$$

is a diffeomorphism provided $v_q \in N_q^\Delta$ and $|v_q| < r_0$; take $V_1$ to be the union of these open balls of radius $r_0$.

Now we can define the manifold structure of $G_M$. The set $G'$ is given the usual product manifold structure; it has an 'outer' boundary $M \times M \times \{1\}$. In order to attach $G''$ to it as an 'inner' boundary, we consider

$$U_1 := \{(q, v_q, \hbar) : (q, \hbar v_q) \in V_1\},$$

which is an open subset of $TM \times [0, 1]$; indeed, it is the union of $TM \times \{0\}$ and the tube of radius $r_0/\hbar$ around $\Delta(M) \times \{\hbar\}$ for each $\hbar \in (0, 1]$. Therefore, the map $\Phi \colon U_1 \to G_M$ given by

$$\begin{aligned}
\Phi(q, v_q, \hbar) &:= \left(\exp_q\left(\tfrac{1}{2}\hbar v_q\right), \exp_q\left(-\tfrac{1}{2}\hbar v_q\right), \hbar\right) &&\text{for } \hbar > 0, \\
\Phi(q, v_q, 0) &:= (q, v_q) &&\text{for } \hbar = 0,
\end{aligned} \tag{6.7}$$

is one-to-one and maps the boundary of $U_1$ onto $G''$. The restriction of $\Phi$ to $U_1' := \{(q, v_q, \hbar) \in U_1 : 0 < \hbar \leq 1\} \subset TM \times (0, 1]$ is a local diffeomorphism between $U_1'$ and its image $U_2' \subset M \times M \times (0, 1]$. One checks that changes of charts are smooth; thus, even if $M$ is not compact, we can construct maps (6.7) locally and patch them together to transport the smooth structure from sets like $U_1$ to the inner boundary of the groupoid $G_M$.

To prove that $G_M$ is a smooth groupoid, one also has to check the required properties of the inclusion $U_M \hookrightarrow G_M$, the maps $r$ and $s$, the inversion and the product. Actually, the present construction is a particular case of the *tangent groupoid to a given groupoid* given by [114] and [161]. They remark that, given a smooth groupoid $G \rightrightarrows U$ (in our case the pair groupoid of $M$), then if $N$ is the normal bundle to $U$ in $G$, the set $N \times \{0\} \uplus G \times (0, 1]$ is a smooth groupoid $\Gamma_U^G$ with diagonal $U \times [0, 1]$, the construction (and therefore the correspondence $M \mapsto G_M$) being functorial. The smoothness properties are proved by repeated application of the following elementary result: if $X$, $X'$ be smooth manifolds and $Y$, $Y'$ respective closed submanifolds, and if $f \colon X \to X'$ a smooth map such that $f(Y) \subset Y'$, then the induced map from $\Gamma_Y^X$ to $\Gamma_{Y'}^{X'}$ is smooth. The detailed proof is given in [22]. See also [147], [148].

## 6.4 Moyal quantization as a continuity condition

We now apply the Gelfand–Naĭmark cofunctor to tangent groupoids. A function on $G_M$ is first of all a pair of functions on $G'$ and $G''$ respectively. The first one is essentially a kernel, the second is the inverse Fourier transform of a function on the cotangent bundle $T^*M$. The condition that both match to give a continuous function on $G_M$ is precisely the quantization rule. For clarity, we consider first the case $M = \mathbb{R}^n$. Let $a(x, \xi)$ be a function on $T^*\mathbb{R}^n$. Its inverse Fourier transform is a function on $T\mathbb{R}^n$:

$$\mathcal{F}^{-1}a(q, v) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi v} a(q, \xi) \, d\xi.$$

On $\mathbb{R}^n$, the exponentials are given by

$$x := \exp_q\left(\tfrac{1}{2}\hbar v\right) = q + \tfrac{1}{2}\hbar v; \quad y := \exp_q\left(-\tfrac{1}{2}\hbar v\right) = q - \tfrac{1}{2}\hbar v. \qquad (6.8)$$

Thus we can solve for $(q, v)$:

$$q = \frac{x+y}{2}; \quad v = \frac{x-y}{\hbar}. \qquad (6.9)$$

To the function $a$ we associate the following family of kernels:

$$k_a(x, y; \hbar) := \hbar^{-n} \mathcal{F}^{-1} a(q, v) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} a\left(\frac{x+y}{2}, \xi\right) e^{i(x-y)\xi/\hbar}\, d\xi,$$

which is just the Moyal quantization formula (6.6). The factor $\hbar^{-n}$ is the Jacobian of the transformation (6.9).

We get the dequantization rule by Fourier inversion:

$$a(q, \xi) = \int_{\mathbb{R}^n} k_a(q + \tfrac{1}{2}\hbar v, q - \tfrac{1}{2}\hbar v)\, e^{iv\xi}\, dv.$$

Here $k_a$ is the kernel of the operator $Q(a)$ of (6.2).

**The general Moyal asymptotic morphism.** If $M$ is a Riemannian manifold, we can now quantize any function $a$ on $T^*M$ such that $\mathcal{F}^{-1}a$ is smooth and has compact support, say $K_a$. For $\hbar_0$ small enough, the map $\Phi$ of (6.7) is defined on $K_a \times [0, \hbar_0)$. For $\hbar < \hbar_0$, the formulas (6.8), (6.9) must be generalized to a transformation between $TM$ and $M \times M \times \{\hbar\}$ whose Jacobian must be determined. We follow the treatment in [145]; see also [22] and [147].

Let $\gamma_{q,v}$ be the geodesic on $M$ starting at $q$ with velocity $v$, with an affine parameter $s$, i.e., $\gamma_{q,tv}(s) \equiv \gamma_{q,v}(ts)$. Locally, we may write

$$\begin{aligned} x &:= \gamma_{q,v}(s), \\ y &:= \gamma_{q,v}(-s), \end{aligned} \quad \text{with Jacobian matrix} \quad \frac{\partial(x, y)}{\partial(q, v)}(s). \qquad (6.10)$$

The Jacobian can be computed from the equations of geodesic deviation [145]. Introduce

$$J(q, v; s) := s^{-n} \frac{\sqrt{\det g(\gamma_{q,v}(s))}\sqrt{\det g(\gamma_{q,v}(-s))}}{\det g(q)} \left| \frac{\partial(x, y)}{\partial(q, v)} \right|(s).$$

This yields a change-of-variables formula:

$$\int_{M \times M} F(x, y)\, \Omega(x)\, \Omega(y) = \int_M \int_{T_q M} F\left(\gamma_{q,v}\left(\tfrac{1}{2}\right), \gamma_{q,v}\left(-\tfrac{1}{2}\right)\right) J\left(q, v; \tfrac{1}{2}\right) \mu_q(v)\, \Omega(q).$$

The general quantization/dequantization recipes are now given by

$$k_a(x, y; \hbar) := \hbar^{-n} J^{-1/2}(q, v, \tfrac{1}{2}\hbar)\, \mathcal{F}^{-1} a(q, v),$$

$$a(q, \xi) = \mathcal{F}\left[J^{1/2}(q, v, \tfrac{1}{2}\hbar)\, k_a(x, y; \hbar)\right],$$

where $(x, y)$ and $(q, v)$ are related by (6.10) with $s = \tfrac{1}{2}\hbar$.

One can check that

$$J(q, v, \tfrac{1}{2}\hbar) = 1 + O(\hbar^2);$$

a long but straightforward computation then shows that we have defined an asymptotic morphism from $C_c^\infty(T^*M)$ to $\mathcal{K}(L^2(M))$. Moreover the tracial property (6.3b) for the associated quantization rule is satisfied:

$$\mathrm{Tr}\big[T_\hbar(a)T_\hbar(b)\big] = \int_{T^*M} a(u)b(u)\,\mu(u).$$

The corresponding $K$-theory map $T_*\colon K^0(T^*M) \to \mathbb{Z}$ is an *analytical index map*; it coincides [161] with the Atiyah–Singer analytical index map [5].

## 6.5  The hexagon and the analytical index

An essentially equivalent argument is done by Connes in $C^*$-algebraic language [37, II.5]. Indeed, given a smooth groupoid $G = G' \uplus G''$ which is a disjoint union of two smooth groupoids with $G'$ open and $G''$ closed in $G$, there is a short exact sequence of $C^*$-algebras

$$0 \longrightarrow C^*(G') \longrightarrow C^*(G) \xrightarrow{\sigma} C^*(G'') \longrightarrow 0$$

where $\sigma$ is the homomorphism defined by restriction from $C_c^\infty(G)$ to $C_c^\infty(G'')$: it is enough to notice that $\sigma$ is continuous for the $C^*$-norms because one takes the supremum of $\|\pi_u(a)\|$ over the closed subset $u \in U''$, and it is clear that $\ker \sigma \simeq C^*(G')$.

There is a short exact sequence of $C^*$-algebras

$$0 \longrightarrow C_0(0, 1] \otimes \mathcal{K} \longrightarrow C^*(G_M) \xrightarrow{\sigma} C_0(T^*M) \longrightarrow 0$$

that yields isomorphisms in $K$-theory:

$$K_j(C^*(G_M)) \xrightarrow{\sigma_*} K_j(C_0(T^*M)) = K^j(T^*M), \quad j = 0, 1. \qquad (6.11)$$

This is seen as follows: since $C^*(M \times M) = \mathcal{K}$, the $C^*$-algebra $C^*(G')$, obtained by completing the (algebraic) tensor product $C_c^\infty(0, 1] \otimes C_c^\infty(M \times M)$, is $C_0(0, 1] \otimes \mathcal{K}$, which is contractible, via the homotopy $\alpha_t(f \otimes A) := f(t \cdot) \otimes A$ for $f \in C_0(0, 1]$, $0 \le t \le 1$; in particular, $K_j(C_0(0, 1] \otimes \mathcal{K}) = 0$. Next, we appeal to the six-term cyclic exact sequence in $K$-theory of $C^*$-algebras [189]:

$$0 = K_0(C_0(0, 1] \otimes \mathcal{K}) \longrightarrow K_0(C^*(G_M)) \xrightarrow{\sigma_*} K_0(C_0(T^*M))$$

$$\delta_1 \big\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big\downarrow \delta_0$$

$$K_1(C_0(T^*M)) \xleftarrow{\sigma_*} K_1(C^*(G_M)) \longleftarrow K_1(C_0(0, 1] \otimes \mathcal{K}) = 0$$

The two trivial groups break the circuit and leave the two isomorphisms (6.11).

The restriction of elements of $C^*(G)$ to the outer boundary $M \times M \times \{1\}$ gives a homomorphism

$$\rho \colon C^*(G_M) \to C^*(M \times M \times \{1\}) \simeq \mathcal{K},$$

and in $K$-theory this yields a homomorphism $\rho_* \colon K_0(C^*(G_M)) \to K_0(\mathcal{K}) = \mathbb{Z}$. The composition $\rho_*(\sigma_*)^{-1} \colon K^0(T^*M) \to \mathbb{Z}$ is just the analytical index map.

## 6.6  Quantization and the index theorem

From the point of view of quantization theory, this is not the whole story. Certainly, in the previous argument, we could have substituted any interval $[0, \hbar_0]$ for $[0, 1]$. But for a given value of $\hbar_0$, not every reasonable function on $T^*M$ can be successfully quantized. For exponential manifolds, like flat phase space or the Poincaré disk, everything should work fine. However, when one tries to apply a similar procedure to compact symplectic manifolds, one typically finds cohomological obstructions. This is dealt with in [83], leading back to the standard results in geometric quantization *à la* Kostant–Kirillov–Souriau. We are left with the impression that the role of the apparatus of Moyal quantization in the foundations of noncommutative (topology and) geometry cannot be fortuitous.

Conversely, one can ask what noncommutative geometry can do for quantization theory. One has first to agree on the meaning (at least, on the mathematical side) of the word 'quantization'. The nearly perfect match afforded by the Moyal machinery in its particular realm is not to be expected in general. For any symplectic manifold, indeed for any Poisson manifold [132], some kind of 'quantum' deformation or star-product can always be found. However, in general a Moyal quantizer is missing.

The modern temperament – see, for instance, [88] – is to consider that *quantization is embodied in the index theorem*. This dictum goes well with the original meaning of the word 'quantization' in Bohr's old quantum theory. In the rare instances where it works well, it appears to give much more information than just a few integers (the indices of a certain Fredholm operator). Two of the more successful examples of quantization are Moyal quantization of finite-dimensional symplectic vector spaces and Kirillov–Kostant–Souriau geometric quantization of flag manifolds. In the latter case, Vergne [218] (see also [14]) has suggested to replace the concept of polarization, central to Kostant's work, by the use of Dirac-type operators – which of course takes us back to the spectral triples of noncommutative geometry.

The new scheme for quantization runs more or less as follows. Let $M$ be a smooth manifold endowed with a spin$^c$ structure (starting from a symplectic manifold, one can introduce a compatible almost complex structure in order to produce the spin$^c$ structure; the results only depend weakly of the choice made, since the set of almost complex structures is contractible). Construct prequantum line bundles $L$ over $M$ according to the KKS recipe, or some improved version like that of [188]. Let $D_L$ be a twisted

Dirac operator for $L$. A quantization of $M$ is the (virtual) Hilbert space

$$\mathcal{H}_{D,L} := \ker D_L^+ - \ker D_L^-.$$

The index theorem gives precisely the (virtual) dimension of such a space. Under favourable circumstances, we can do better. In the case of flag manifolds, one quantizes $G$-bundles and obtains $G$-Hilbert spaces. Then the $G$-index theorem gives us the character of the Kirillov representation associated to $L$ [12], which contains all the quantum information we seek.

Moyal quantization, on the other hand, is a tool of choice for the proof of the index theorem, as indicated here. The 'logical' (though not the historical) way to go about the Index Theorem would be to prove it first in the flat case using Moyal quantum mechanics – see [76] or [83] – and then go to analytically simpler but geometrically more involved cases. Conversely, one is left with the problem of how to recover the whole of Moyal theory from the index theorem.

The tangent groupoid construction, which we have given here for compact boundaryless manifolds, has been extended to manifolds with boundary in [1], where the $K$-theory of an appropriate symbol algebra is also computed.

# 7

# Equivalence of geometries

We wish to classify spin geometries and to get some idea of how many geometries of a given type are available to us. When modelling physical systems that have an underlying geometry, we naturally wish to select the most suitable geometry from several plausible candidates. The first question to ask, then, is: when are two spin geometries the same?

## 7.1 Unitary equivalence of spin geometries

In order to compare two spin geometries $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$ and $(\mathcal{A}', \mathcal{H}', D'; \Gamma', J')$, we focus first of all on the algebras $\mathcal{A}$ and $\mathcal{A}'$. It is natural to require that these be isomorphic, that is to say, that there be an involutive isomorphism $\phi \colon A \to A'$ between their $C^*$-algebra closures, such that $\phi(\mathcal{A}) = \mathcal{A}'$. Since these algebras define spin geometries only through their representations on the Hilbert spaces, we lose nothing by assuming that they are the *same* algebra $\mathcal{A}$. We can also assume that the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ are the same, so that $\mathcal{A}$ acts on $\mathcal{H}$ with two possibly different faithful representations.

One must then match the operator pairs $D$ and $D'$, etc., on the Hilbert space $\mathcal{H}$. We are thus led to the notion of *unitary* equivalence of spin geometries.

**Definition 16.** Two spin geometries $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$ and $(\mathcal{A}, \mathcal{H}, D'; \Gamma', J')$ with the same algebra and Hilbert space are *unitarily equivalent* if there is a unitary operator $U \colon \mathcal{H} \to \mathcal{H}$ such that

(a) $UD = D'U$, $U\Gamma = \Gamma'U$ and $UJ = J'U$;

(b) $U\pi(a)U^{-1} \equiv \pi(\sigma(a))$ for an automorphism $\sigma$ of $\mathcal{A}$.

By 'automorphism of $\mathcal{A}$' is meant a $*$-automorphism of the $C^*$-algebra $A$ that maps $\mathcal{A}$ into itself. If $J' = J$, we also get

$$U\pi^\circ(b)U^{-1} = UJ\pi(b^*)J^{-1}U^{-1} = J\pi(\sigma(b^*))J^{-1} = J\pi(\sigma(b)^*)J^{-1} = \pi^\circ(\sigma(b)).$$

To be sure that this definition is consistent, let us check the following statement: given a spin geometry $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$ and a unitary operator on $\mathcal{H}$ such that $U\pi(\mathcal{A})U^{-1} = \pi(\mathcal{A})$, then $(\mathcal{A}, \mathcal{H}, UDU^{-1}; U\Gamma U^{-1}, UJU^{-1})$ is also a spin geometry.

Firstly, $\pi(a) \mapsto U\pi(a)U^{-1} =: \pi(\sigma(a))$ determines an automorphism $\sigma$ of $\mathcal{A}$, since $\pi$ is faithful.

The operator $D' := UDU^{-1}$ has the same spectrum as $D$, so the dimension is unchanged and Poincaré duality remains valid. The first-order condition is satisfied, since

$$[[D', \sigma(a)], \sigma(b)^\circ] = U[[D, a], b^\circ]U^{-1} = 0.$$

Also, $U\pi_D(c)U^{-1} = \pi_{D'}(\sigma(c))$ for $c \in C_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$, where the action of $\sigma$ on $\mathcal{A}$ and $\mathcal{A}^\circ$ is extended to Hochschild cochains in the obvious way. In particular, if $c$ is the orientation cycle, then

$$\pi_{D'}(\sigma(c)) = U\Gamma U^{-1} = \Gamma \quad \text{or} \quad \pi_{D'}(\sigma(c)) = UU^{-1} = 1,$$

according as the dimension is even or odd. Thus $\sigma(c)$ is the orientation cycle for $(\mathcal{A}, \mathcal{H}, UDU^{-1}; U\Gamma U^{-1}, UJU^{-1})$, too.

For the finiteness property, the space of smooth vectors $\mathcal{H}'_\infty = \bigcap_k \mathrm{Dom}(D')^k$ equals $U\mathcal{H}^\infty$, and we may define $(U\xi \mid U\eta)' := \sigma(\xi \mid \eta)$ for $\xi, \eta \in \mathcal{H}^\infty$. This is the appropriate hermitian pairing on $\mathcal{H}'_\infty$, since (3.7) shows that

$$\fint \sigma(a)\,(U\xi \mid U\eta)'\,|D'|^{-n} = \fint Ua(\xi \mid \eta)U^{-1}|D'|^{-n} = \fint Ua(\xi \mid \eta)|D|^{-n}U^{-1}$$

$$= \fint a(\xi \mid \eta)|D|^{-n} = \langle \eta \mid a\xi \rangle = \langle U\eta \mid \sigma(a)U\xi \rangle.$$

**Unitary equivalence of toral geometries.** Let us now consider the effect of the hyperbolic automorphism (4.6) of the algebra $\mathcal{A}_\theta$ on the spin geometry $\mathbb{T}^2_{\theta,\tau}$ constructed in Chapter 4.

The mapping $\underline{a} \mapsto \underline{\sigma(a)}$ determined by $\sigma(u) := u^a v^b$, $\sigma(v) := u^c v^d$ extends to a unitary operator $U_\sigma$ on $\mathcal{H}_0 = L^2(\mathcal{A}_\theta, \tau)$, since it just permutes the orthonormal basis $\{\underline{u^m v^n} : m, n \in \mathbb{Z}\}$ (while each basis vector is also multiplied by a phase factor of absolute value 1). Let $U = U_\sigma \oplus U_\sigma$ be the corresponding unitary operator on $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$; it is evident that $U\pi(a)U^{-1} = \pi(\sigma(a))$ for $a \in \mathcal{A}_\theta$. By construction, $U\Gamma = \Gamma U$. Also, $UJ = JU$ on account of $U_\sigma J_0 \underline{a} = U_\sigma \underline{a^*} = \underline{\sigma(a)^*} = J_0 U_\sigma \underline{a}$ because $\sigma$ is a $*$-isomorphism.

The inverse-length operator transforms as

$$D_\tau = -i \begin{pmatrix} 0 & -\partial_\tau^\dagger \\ \underline{\partial_\tau} & 0 \end{pmatrix} \longmapsto UD_\tau U^{-1} = -i \begin{pmatrix} 0 & -\tilde{\partial}_\tau^\dagger \\ \underline{\tilde{\partial}_\tau} & 0 \end{pmatrix}$$

where $\underline{\tilde{\partial}_\tau} = U_\sigma \underline{\partial_\tau} U_\sigma^{-1}$ corresponds to $\tilde{\partial}_\tau = \sigma \circ \partial_\tau \circ \sigma^{-1}$ on $\mathcal{A}_\theta$. Since

$$\sigma^{-1}(u) = \lambda^{bd(a-c-1)/2} u^d v^{-b}, \quad \sigma^{-1}(v) = \lambda^{ac(d-b-1)/2} u^{-c} v^a,$$

we get at once $\tilde{\partial}_\tau u = 2\pi i(d - b\tau)u$, $\tilde{\partial}_\tau v = 2\pi i(a\tau - c)v$. Since $\partial_\tau = \delta_1 + \tau\delta_2$, we arrive at

$$\tilde{\partial}_\tau = (d - b\tau)\,\delta_1 + (a\tau - c)\,\delta_2.$$

This is tantamount to replacing $\tau$ by $\sigma^{-1} \rhd \tau = (a\tau - c)/(d - b\tau)$ in the definition of $D_\tau$, together with a rescaling in order to preserve the area given by the orientation cycle (4.12). It is easy to check that $\langle\phi, c\rangle = \langle\phi, \sigma(c)\rangle$ owing to $ad - bc = 1$. Here $\sigma^{-1} \rhd \tau$ denotes the action of the element $\sigma^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ on the upper half-plane by Möbius transformations. Notice that the elliptic curves with period pairs $(1, \tau)$ and $(d - b\tau, a\tau - c)$ have period parallelograms of the same area.

Thus, for each spin geometry $\mathbb{T}^2_{\theta,\tau}$, there is a family of unitarily equivalent geometries $\mathbb{T}^2_{\theta,\sigma^{-1}\rhd\tau}$, for $\sigma \in \mathrm{SL}(2, \mathbb{Z})$. We thereby obtain a family of spin geometries over $\mathcal{A}_\theta$, parametrized up to unitary equivalence by the fundamental domain of the modular group $\mathrm{PSL}(2, \mathbb{Z})$.

**Action of inner automorphisms.** On any spin geometry $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$, there is an action by inner automorphisms of the algebra $\mathcal{A}$. If $u$ is a unitary element of the algebra $\mathcal{A}$, i.e., $u^*u = uu^* = 1$, consider the unitary operator on $\mathcal{H}$ given by

$$U := \pi(u)\pi^\circ(u^{-1}) = uJuJ^\dagger = JuJ^\dagger u.$$

If $a \in \mathcal{A}$, then $UaU^{-1} = uau^{-1}$ since $JuJ^\dagger$ commutes with $a$, so $U$ implements the inner automorphism $\sigma_u(a) := u\,a\,u^{-1}$. Next, $J^2 = \pm1$ implies that

$$UJ = uJu = \pm uJ^\dagger u = J^2 uJ^\dagger u = JU.$$

In the even case, $\Gamma$ commutes with $\pi(u)$ and $\pi^\circ(u^{-1})$, so that $U\Gamma = \Gamma U$, too.

The given geometry is thus unitarily equivalent to $(\mathcal{A}, \mathcal{H}, {}^u D; \Gamma, J)$, where

$$\begin{aligned}
{}^u D &:= UDU^* = JuJ^\dagger uDu^*Ju^*J^\dagger = JuJ^\dagger(D + u\,[D, u^*])Ju^*J^\dagger \\
&= JuJ^\dagger DJu^*J^\dagger + u\,[D, u^*] = D + JuJ^\dagger[D, Ju^*J^\dagger] + u\,[D, u^*] \qquad (7.1) \\
&= D + u\,[D, u^*] \pm Ju\,[D, u^*]J^\dagger.
\end{aligned}$$

Here we have used the first-order condition and the relation $JD = \pm DJ$; the latter gives the $\pm$ sign on the right hand side, which is negative iff $n \equiv 1 \bmod 4$. Notice that the operator $u[D, u^*] = uDu^* - D$ is bounded and selfadjoint in $\mathcal{L}(\mathcal{H})$.

## 7.2 Morita equivalence and connections

The unitary equivalence of spin geometries helps to eliminate obvious redundancies, but it is by no means the only way to compare geometries. For one thing, the metric

distance (1.8) is unchanged – if we think of the right hand side of (1.8) as defining the distance between pure states $\hat{p}, \hat{q}$ of the algebra $\mathcal{A}$.

We need a looser notion of equivalence between spin geometries that allows to vary not just the operator data but also the algebra and the Hilbert space. Here the Morita equivalence of algebras gives us a clue as to how to proceed. We can change the algebra $\mathcal{A}$ to a Morita-equivalent algebra $\mathcal{B}$, which also involves changing the representation space according to well-defined rules. How should we then adapt the remaining data $\Gamma$, $J$ and $D$, in order to obtain a *Morita equivalence of spin geometries*?

We start with any spin geometry $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$ and a finitely generated projective right $\mathcal{A}$-module $\mathcal{E}$ (recall that we are assuming that the algebra $\mathcal{A}$ is unital.) Using the representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and the antirepresentation $\pi^\circ : b \mapsto J\pi(b^*)J^\dagger$, we can regard the space $\mathcal{H}$ as an $\mathcal{A}$-bimodule. This allows us to introduce the vector space

$$\widetilde{\mathcal{H}} := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \bar{\mathcal{E}}. \tag{7.2}$$

If $\mathcal{E} = p\mathcal{A}^m$, then $\bar{\mathcal{E}} = \bar{\mathcal{A}}^m p$ and $\widetilde{\mathcal{H}} = \pi(p)\pi^\circ(p)[\mathcal{H} \otimes M_m(\mathbb{C})]$, so that $\widetilde{\mathcal{H}}$ becomes a Hilbert space under the scalar product

$$\langle r \otimes \eta \otimes \bar{q} \mid s \otimes \xi \otimes \bar{t} \rangle := \langle \eta \mid \pi(r \mid s)\,\pi^\circ(t \mid q)\,\xi \rangle.$$

If $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is $\mathbb{Z}_2$-graded, there is a corresponding $\mathbb{Z}_2$-grading of $\widetilde{\mathcal{H}}$. The antilinear correspondence $s \mapsto \bar{s}$ between $\mathcal{E}$ and $\bar{\mathcal{E}}$ also gives an obvious way to amplify $J$ to $\widetilde{\mathcal{H}}$:

$$\widetilde{J}(s \otimes \xi \otimes \bar{t}) := t \otimes J\xi \otimes \bar{s}. \tag{7.3}$$

Let $\mathcal{B} = \mathcal{E} \otimes_{\mathcal{A}} \bar{\mathcal{E}}$ be the algebra of ket-bras (1.2); recall that $\mathcal{E}$ is a left $\mathcal{B}$-module and the action of $\mathcal{B}$ on $\mathcal{E}$ commutes with the right action of $\mathcal{A}$. Then

$$\rho(b) : s \otimes \xi \otimes \bar{t} \longmapsto b\,s \otimes \xi \otimes \bar{t}$$

yields a representation $\rho$ of $\mathcal{B}$ on $\widetilde{\mathcal{H}}$, and an antirepresentation

$$\rho^\circ(b) := \widetilde{J}\rho(b^*)\widetilde{J}^\dagger : s \otimes \xi \otimes \bar{t} \longmapsto s \otimes \xi \otimes \bar{t}\,b,$$

where $\bar{t}\,b := \overline{b^*t}$, of course. The actions $\rho$, $\rho^\circ$ of $\mathcal{B}$ on $\widetilde{\mathcal{H}}$ obviously commute.

**Where connections come from.** The nontrivial part of the construction of the new spin geometries $(\mathcal{B}, \widetilde{\mathcal{H}}, \widetilde{D}; \widetilde{\Gamma}, \widetilde{J})$ is the determination of an appropriate operator $\widetilde{D}$ on $\widetilde{\mathcal{H}}$. Guided by the differential properties of Dirac operators, the most suitable procedure is to postulate a *Leibniz rule*:

$$\widetilde{D}(s \otimes \xi \otimes \bar{t}) := (\nabla s)\xi \otimes \bar{t} + s \otimes D\xi \otimes \bar{t} + s \otimes \xi(\overline{\nabla t}), \tag{7.4}$$

where $\nabla s$, $\nabla t$ belong to some space whose elements can be represented on $\mathcal{H}$ by suitable extensions of $\pi$ and $\pi^\circ$.

Consistency of (7.4) with the actions of $\mathcal{A}$ on $\mathcal{E}$ and $\mathcal{H}$ demands that $\nabla$ itself comply with a Leibniz rule. Since $sa \otimes \xi \otimes \bar{t} = s \otimes a\xi \otimes \bar{t}$ for $a \in \mathcal{A}$, (7.4) entails

$$\nabla(sa)\xi \otimes \bar{t} + s \otimes a\, D\xi \otimes \bar{t} = (\nabla s)a\xi \otimes \bar{t} + s \otimes D\, a\xi \otimes \bar{t},$$

so we infer that

$$\nabla(sa) = (\nabla s)\, a + s \otimes [D, a], \tag{7.5}$$

or more precisely, $\nabla(sa) = (\nabla s)\pi(a) + s \otimes [D, \pi(a)]$ as operators on $\mathcal{H}$.

To satisfy these requirements, we introduce the space of bounded operators

$$\Omega_D^1 := \operatorname{span}\{a\,[D, b] : a, b \in \mathcal{A}\} \subseteq \mathcal{L}(\mathcal{H}),$$

which is evidently an $\mathcal{A}$-bimodule under the actions $c \triangleright a\,[D, b] := ca\,[D, bc]$ and $a\,[D, b]\triangleleft c := a\,[D, bc] - ab\,[D, c]$. The notation $\Omega_D^1$ is chosen to remind us of differential 1-forms; for the commutative spin geometry $(C^\infty(M), L^2(M, S), D\!\!\!/; \gamma(\chi), J)$, we indeed get

$$\Omega_{D\!\!\!/}^1 = \{\gamma(\alpha) : \alpha \in \mathcal{A}^1(M)\},$$

i.e., conventional 1-forms on $M$, represented on spinor space as Clifford multiplication operators.

**Definition 17.** We can now form the right $\mathcal{A}$-module $\mathcal{E} \otimes_\mathcal{A} \Omega_D^1$. A *connection* on $\mathcal{E}$ is a linear mapping

$$\nabla \colon \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} \Omega_D^1$$

that satisfies the Leibniz rule (7.5).

It is worth mentioning that only *projective* modules admit connections [58]. In the present case, if we define linear maps

$$0 \longrightarrow \mathcal{E} \otimes_\mathcal{A} \Omega_D^1 \overset{j}{\longrightarrow} \mathcal{E} \otimes_\mathbb{C} \mathcal{A} \overset{m}{\longrightarrow} \mathcal{E} \longrightarrow 0$$

by $j(s\,[D, a]) := sa \otimes 1 - s \otimes a$ and $m(s \otimes a) := sa$, we get a short exact sequence of right $\mathcal{A}$-modules (think of $\mathcal{E} \otimes_\mathbb{C} \mathcal{A}$ as a free $\mathcal{A}$-module generated by a vector-space basis of $\mathcal{E}$). Any linear map $\nabla \colon \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} \Omega_D^1$ gives a linear section of $m$ by $f(s) := s \otimes 1 - j(\nabla s)$. Then $f(sa) - f(s)a = j(s\,[D, a] - \nabla(sa) + (\nabla s)a)$, so $f$ is an $\mathcal{A}$-module map precisely when $\nabla$ satisfies the Leibniz rule (7.5). If that happens, $f$ splits the exact sequence and embeds $\mathcal{E}$ as a direct summand of the free $\mathcal{A}$-module $\mathcal{E} \otimes_\mathbb{C} \mathcal{A}$, so $\mathcal{E}$ is projective.

**Hermitian connections.** The operator $\widetilde{D}$ must be selfadjoint on $\widetilde{\mathcal{H}}$. If $\xi, \eta \in \mathrm{Dom}(D)$, we get

$$\langle r \otimes \eta \otimes \bar{q} \mid \widetilde{D}(s \otimes \xi \otimes \bar{t}) \rangle = \langle \eta \mid \pi_D(r \mid \nabla s)\, \pi^\circ(t \mid q)\, \xi \rangle$$
$$+ \langle \eta \mid \pi(r \mid s)\, \pi^\circ(t \mid q)\, D\xi \rangle$$
$$+ \langle \eta \mid \pi(r \mid s)\, \pi^\circ_D(\nabla t \mid q)\, \xi \rangle,$$
$$\langle \widetilde{D}(r \otimes \eta \otimes \bar{q}) \mid s \otimes \xi \otimes \bar{t} \rangle = \langle \eta \mid \pi_D(\nabla r \mid s)\, \pi^\circ(t \mid q)\, \xi \rangle$$
$$+ \langle D\eta \mid \pi(r \mid s)\, \pi^\circ(t \mid q)\, \xi \rangle$$
$$+ \langle \eta \mid \pi(r \mid s)\, \pi^\circ_D(t \mid \nabla q)\, \xi \rangle.$$

This reduces to the condition that

$$(r \mid \nabla s) - (\nabla r \mid s) = [D, (r \mid s)] \quad \text{for all } r, s \in \mathcal{E}. \tag{7.6}$$

where the first-order condition ensures commutation of $\pi_D(\Omega^1_D)$ with $\pi^\circ(\mathcal{A})$.

We say that the connection $\nabla$ is *Hermitian* (with respect to $D$) if (7.6) holds. The minus sign is due to the presence of the selfadjoint operator $D$, while a skewadjoint differential operator is used in the standard definition of a metric-preserving connection [12], [149].

To sum up: two *spin geometries* $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$ and $(\mathcal{B}, \widetilde{\mathcal{H}}, \widetilde{D}; \widetilde{\Gamma}, \widetilde{J})$ are *Morita equivalent* if there exist a finitely generated projective right $\mathcal{A}$-module $\mathcal{E}$ and an $\Omega^1_D$-valued Hermitian connection $\nabla$ on $\mathcal{E}$, such that: $\mathcal{B} = \mathcal{E} \otimes_{\mathcal{A}} \bar{\mathcal{E}}$, $\widetilde{\mathcal{H}}$ and $\widetilde{\Gamma}$ are given by (7.2), $\widetilde{J}$ by (7.3), and $\widetilde{D}$ by (7.4).

## 7.3 Vector bundles over noncommutative tori

The finitely generated projective modules over the irrational rotation $C^*$-algebra $A_\theta$ were defined in [32] and fully classified in [182]. (Actually, [182] also constructs vector bundles over $\mathbb{T}^2$ that represent distinct classes in $K_0(C^\infty(\mathbb{T}^2)) \simeq \mathbb{Z} \oplus \mathbb{Z}$; but the projective modules so obtained are unlike those of the irrational case.) To describe the latter, we return to the Weyl operators (4.1). The translation and multiplication operators $W_\theta(a, 0)$ and $W_\theta(0, b)$ are generated by $i\, d/dt$ and $t$; the space of smooth vectors for these derivations is the Schwartz space $\mathcal{S}(\mathbb{R})$. Clearly $\mathcal{S}(\mathbb{R})$ is a left $\mathcal{A}_\theta$-module under the action given by

$$u \triangleright \psi = W_\theta(1, 0)\psi : t \mapsto \psi(t - 1)$$
$$v \triangleright \psi = W_\theta(0, 1)\psi : t \mapsto e^{2\pi i \theta t}\, \psi(t).$$

But it can also be made a *right* $\mathcal{A}_\theta$-module by making the generators act in another way. If $p$ is any integer, we can define

$$\psi \triangleleft u := W_\theta(p - \theta, 0)\psi : t \mapsto \psi(t - p + \theta),$$

$$\psi \triangleleft v := W_\theta(0, 1/\theta)\psi : t \mapsto e^{2\pi i t} \psi(t).$$

Therefore

$$\psi \triangleleft vu = e^{-\pi i(p-\theta)} W_\theta(p - \theta, 1/\theta)\psi, \quad \psi \triangleleft uv = e^{+\pi i(p-\theta)} W_\theta(p - \theta, 1/\theta)\psi,$$

thus $\psi \triangleleft vu = e^{2\pi i\theta} \psi \triangleleft uv$. Since the generators act compatibly with the commutation relation (4.3), this defines a right action of $\mathcal{A}_\theta$ on $\mathcal{S}(\mathbb{R})$. This right module will be denoted $\mathcal{E}_p$.

One can define more $\mathcal{A}_\theta$-modules by a simple trick. Let $q$ be a positive integer; the Weyl operators act on $\mathcal{S}(\mathbb{R}^q) = \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^q$ as $W_\theta(a, b) \otimes 1_q$. If $z \in M_q(\mathbb{C})$ is the cyclic shift $(x_1, \ldots, x_q) \mapsto (x_2, \ldots, x_q, x_1)$ and $w \in M_q(\mathbb{C})$ is the diagonal operator $(x_1, \ldots, x_q) \mapsto (\zeta x_1, \zeta^2 x_2, \ldots, x_q)$ for $\zeta := e^{2\pi i/q}$, then $zw = e^{2\pi i/q} wz$, so that the maps

$$\psi \triangleleft u := \left(W_\theta(\tfrac{p}{q} - \theta, 0) \otimes z^p\right)\psi, \quad \psi \triangleleft v := \left(W_\theta(0, \tfrac{1}{\theta}) \otimes w\right)\psi$$

satisfy $\psi \triangleleft vu = \lambda \psi \triangleleft uv$ with $\lambda = e^{2\pi i(\theta - p/q)} e^{2\pi i p/q} = e^{2\pi i\theta}$. This right action of $\mathcal{A}_\theta$ on $\mathcal{S}(\mathbb{R}^q)$ defines a right module $\mathcal{E}_{p,q}$.

It turns out that the free modules $\mathcal{A}_\theta^m$ and these $\mathcal{E}_{p,q}$ (with $p, q \in \mathbb{Z}$, $q > 0$) are mutually nonisomorphic and any finitely generated projective right $\mathcal{A}_\theta$-module is isomorphic to one of them. It is perhaps not obvious that the $\mathcal{E}_{p,q}$ are finitely generated and projective. This is proved in [182], using the following Hermitian pairing [55] that makes $\mathcal{E}_{p,q}$ a pre-$C^*$-module over $\mathcal{A}_\theta$:

$$(\phi \mid \psi) := \sum_{r,s} u^r v^s \langle \phi \triangleleft u^r v^s \mid \psi \rangle_{L^2(\mathbb{R}^q)} \quad \text{for } \phi, \psi \in \mathcal{S}(\mathbb{R}^q),$$

where the coefficients, in the case $q = 1$, are:

$$\langle \phi \triangleleft u^r v^s \mid \psi \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} e^{-2\pi i st} \bar{\phi}(t - r(p - \theta))\psi(t)\, dt. \tag{7.7}$$

We shall soon verify projectivity in another way, by introducing connections.

**The Morita equivalent algebras.** To reduce notational complications, we take $q = 1$. The ket-bra algebra $\mathcal{B} := \mathcal{E}_p \otimes_{\mathcal{A}_\theta} \bar{\mathcal{E}}_p$ is generated by Weyl operators that commute with $W_\theta(p - \theta, 0)$ and $W_\theta(0, 1/\theta)$. In view of (4.2), we can take as generators the operators

$$U := W_\theta(1, 0), \quad V := W_\theta(0, 1/\theta(p - \theta)).$$

Then $VU = \mu\, UV$ where $\mu = e^{2\pi i/(p-\theta)}$, so that $\mathcal{B} \simeq \mathcal{A}_{1/(p-\theta)}$.

For the simplest case $p = 0$, $q = 1$, we see that

$$U\psi(t) = \psi(t-1), \quad V\psi(t) = e^{-2\pi i t/\theta} \psi(t), \tag{7.8}$$

so that $\mathcal{A}_\theta$ and $\mathcal{A}_{-1/\theta}$ are Morita equivalent via $\mathcal{E}_0$.

It is known [179] that $\mathcal{A}_\theta$ and $\mathcal{A}_\phi$ are Morita equivalent if and only if either $\phi$ or $-\phi$ lies in the orbit of $\theta$ under the action of $\mathrm{SL}(2, \mathbb{Z})$, i.e., if and only if $\pm\phi = (a\theta + b)/(c\theta + d)$. The proof is $K$-theoretic: since $\tau_*(K_0(\mathcal{A}_\theta)) = \mathbb{Z} + \mathbb{Z}\theta$, a necessary condition is that $\mathbb{Z} + \mathbb{Z}\phi = r(\mathbb{Z} + \mathbb{Z}\theta)$ for some $r > 0$. Sufficiency is proved by exhibiting an appropriate equivalence bimodule $\mathcal{E}_{p,q}$.

## 7.4 Morita-equivalent toral geometries

Let us now construct a Hermitian connection $\nabla$ (with respect to $D_\tau$) on the $\mathcal{A}_\theta$-module $\mathcal{E}_0 = \mathcal{S}(\mathbb{R})$. For that, we must first determine the bimodule $\Omega^1_{D_\tau}$. Clearly

$$\pi(a)\,[D_\tau, \pi(b)] = -i \begin{pmatrix} 0 & -a\,\partial^*_\tau b \\ a\,\partial_\tau b & 0 \end{pmatrix},$$

so that $\Omega^1_{D_\tau} \simeq \mathcal{A}_\theta \oplus \mathcal{A}_\theta$ as $\mathcal{A}_\theta$-bimodules. Thus $\mathcal{E}_0 \otimes_{\mathcal{A}} \Omega^1_{D_\tau} \simeq \mathcal{E}_0 \oplus \mathcal{E}_0$.

Therefore, $\nabla\psi = (\nabla'\psi, \nabla''\psi)$ where $\nabla', \nabla''$ are two derivations on $\mathcal{S}(\mathbb{R})$. The corresponding Leibniz rules are given by (7.5):

$$\nabla'(\psi \triangleleft a) = (\nabla'\psi) \triangleleft a - i\,\psi \triangleleft \partial_\tau a, \quad \nabla''(\psi \triangleleft a) = (\nabla''\psi) \triangleleft a + i\,\psi \triangleleft \partial^*_\tau a.$$

This implies that $\nabla' = -i(\nabla_1 + \tau\,\nabla_2)$ and $\nabla'' = -i(\nabla_1 + \bar\tau\,\nabla_2)$, where $\nabla_1, \nabla_2$ comply with Leibniz rules involving the basic derivations:

$$\nabla_j(\psi \triangleleft a) = (\nabla_j\psi) \triangleleft a + \psi \triangleleft \delta_j a, \quad j = 1, 2,$$

and it is enough to check these relations for $a = u, v$.

It will come as no surprise that $\nabla_1$ and $\nabla_2$ are just the position and momentum operators of quantum mechanics, up to a factor $i\theta/2\pi$; in fact,

$$\nabla_1\psi(t) := -\frac{2\pi i t}{\theta} \psi(t), \quad \nabla_2\psi(t) := \psi'(t). \tag{7.9}$$

One immediately checks that, as required:

$$\nabla_1(\psi \triangleleft u) - (\nabla_1\psi) \triangleleft u = [t \mapsto 2\pi i\,\psi(t + \theta)] = \psi \triangleleft \delta_1 u,$$
$$\nabla_1(\psi \triangleleft v) - (\nabla_1\psi) \triangleleft v = 0 = \psi \triangleleft \delta_1 v,$$
$$\nabla_2(\psi \triangleleft u) - (\nabla_2\psi) \triangleleft u = 0 = \psi \triangleleft \delta_2 u,$$
$$\nabla_2(\psi \triangleleft v) - (\nabla_2\psi) \triangleleft v = [t \mapsto 2\pi i\,e^{2\pi i t}\psi(t)] = \psi \triangleleft \delta_2 v.$$

Thus $\nabla$ is a connection satisfying (7.5) with $D = D_\tau$.

To see that $\nabla$ is Hermitian, it is enough to note that (7.6) reduces to

$$(\phi \mid \nabla'\psi) - (\nabla''\phi \mid \psi) = -i\,\partial_\tau(\phi \mid \psi) \quad \text{for } \phi, \psi \in \mathcal{S}(\mathbb{R}),$$

or equivalently,

$$(\phi \mid \nabla_j\psi) + (\nabla_j\phi \mid \psi) = \delta_j(\phi \mid \psi), \quad j = 1, 2, \tag{7.10}$$

where the $\mathcal{A}_\theta$-valued Hermitian pairing on $\mathcal{S}(\mathbb{R})$ is the special case of (7.7):

$$(\phi \mid \psi) := \sum_{r,s} a_{rs}\, u^r v^s, \quad a_{rs} := \int_{\mathbb{R}} e^{-2\pi i s t}\, \bar{\phi}(t + r\theta)\, \psi(t)\, dt.$$

This can be verified by direct calculation. For instance, when $j = 2$, the left hand side of (7.10) is computed with an integration by parts:

$$\sum_{r,s} u^r v^s \int_{\mathbb{R}} e^{-2\pi i s t} \frac{d}{dt}\big(\bar{\phi}(t + r\theta)\psi(t)\big)\, dt = \sum_{r,s} 2\pi i s\, a_{rs}\, u^r v^s = \delta_2(\phi \mid \psi).$$

**The spin geometry on $\mathcal{A}_{-1/\theta}$.** We take stock of the new spin geometry. Its algebra is $\mathcal{A}_{-1/\theta}$, with generators $U$, $V$ of (7.8). The Hilbert space is $\mathbb{Z}_2$-graded, with $\widetilde{\mathcal{H}}^+ = \mathcal{E}_0 \otimes_\mathcal{A} \mathcal{H}^+ \otimes_\mathcal{A} \bar{\mathcal{E}}_0$, that we can identify with $L^2(\mathcal{A}_{-1/\theta}, \tau)$. Under this identification, $\widetilde{J}$ becomes $\underline{a} \mapsto \underline{a}^*$, as before.

It remains to identify the operator $\widetilde{D}$, whose general form has been determined in Chapter 4. We find that $[\widetilde{D}, U](\psi \otimes \xi \otimes \bar{\phi}) = \big([\nabla, U]\psi\big)\xi \otimes \bar{\phi}$ for $\psi, \phi \in \mathcal{S}(\mathbb{R})$, $\xi \in \mathcal{H}$, where

$$[\nabla, U] = \begin{pmatrix} 0 & [\nabla'', U] \\ [\nabla', U] & 0 \end{pmatrix}.$$

It is immediate from the definitions (7.8), (7.9) that

$$[\nabla_1, U] = -\frac{2\pi i}{\theta} U = -\frac{1}{\theta}\delta_1 U, \quad [\nabla_2, V] = -\frac{2\pi i}{\theta} V = -\frac{1}{\theta}\delta_2 V.$$

and $[\nabla_1, V] = [\nabla_2, U] = 0$. Thus

$$\widetilde{D} = \frac{i}{\theta} \begin{pmatrix} 0 & -\underline{\partial}_\tau^\dagger \\ \underline{\partial}_\tau & 0 \end{pmatrix}.$$

Setting aside the scale factor $-1/\theta$, we see that the modulus $\tau$ is *unchanged*. We conclude that the geometries $\mathbb{T}^2_{\theta,\tau}$ and $\mathbb{T}^2_{-1/\theta,\tau}$ are Morita equivalent.

## 7.5  Gauge potentials

We may ask what Morita equivalence entails when the algebra $\mathcal{A}$ is unchanged, and the equivalence bimodule is $\mathcal{A}$ itself. Regarded as a right $\mathcal{A}$-module, $\mathcal{A}$ carries a standard Hermitian connection with respect to $D$, namely

$$\mathrm{Ad}_D \colon \mathcal{A} \to \Omega_D^1, \quad b \mapsto [D, b].$$

By the Leibniz rule (7.5), any connection differs from $\mathrm{Ad}_D$ by an element of $\Omega_D^1$:

$$\nabla b =: [D, b] + \mathbb{A}b, \tag{7.11}$$

where

$$\mathbb{A} := \sum_j a_j [D, b_j] \quad \text{(finite sum)}$$

lies in $\Omega_D^1$. We call it a *gauge potential* if it is selfadjoint: $\mathbb{A}^* = \mathbb{A}$. Hermiticity of the connection for the pairing $(a \mid b) := a^* b$ demands that $a^* \nabla b - (\nabla a)^* b = [D, a^* b]$, that is, $a^*(\mathbb{A} - \mathbb{A}^*)b = 0$ for all $a, b \in \mathcal{A}$, so a Hermitian connection on $\mathcal{A}$ is indeed given by a gauge potential $\mathbb{A}$.

On substituting the connection (7.11) in the recipe (7.4) for an extended Dirac operator, one obtains

$$\begin{aligned}
\widetilde{D}(b\xi) &= ([D, b] + \mathbb{A}b)\xi + b\,D\xi \pm b\,J(\nabla 1)J^\dagger \xi \\
&= (D + \mathbb{A} \pm J\mathbb{A}J^\dagger)(b\xi),
\end{aligned} \tag{7.12}$$

with the same sign are as in (7.1), that is, negative if and only if $n \equiv 1 \bmod 4$. Therefore, the *gauge transformation*

$$D \longmapsto D + \mathbb{A} \pm J\mathbb{A}J^\dagger$$

yields a spin geometry that is *Morita-equivalent* to the original. Another way of saying this is that the spin geometries whose other data $(\mathcal{A}, \mathcal{H}, \Gamma, J)$ are fixed form an affine space modelled on the selfadjoint part of $\Omega_D^1$.

Notice that the inner automorphisms of (7.1) yield a special case of (7.12), with $\mathbb{A} = u\,[D, u^*]$. These and some more general gauge transformations are discussed in [40], [150], [192], [193] as 'fluctuations of the metric' on an underlying noncommutative manifold.

In summary, we have shown how the classification of spin geometries up to Morita equivalence allows a first-order differential calculus to enter the picture, via the Hermitian connections. In the next section, we shall explore the various spin geometries on a noncommutative manifold from a variational point of view.

# 8

# Action functionals

On a differential manifold, one may use many Riemannian metrics; on a spin manifold with a given Riemannian metric, there may be many distinct (i.e., unitarily inequivalent) spin geometries. An important task, already in the commutative case, is to select, if possible, a particular geometry by some general criterion, such as minimization of an action functional, a time-honoured tradition in physics. In the noncommutative case, the minimizing geometries are often not unique, leading to the phenomenon of spontaneous symmetry breaking [71], an important motivation for physical applications.

## 8.1 Algebra automorphisms and the metric

In this chapter, we fix the data $(\mathcal{A}, \mathcal{H}, \Gamma, J)$ of a spin geometry and consider how the inverse-distance operator $D$ may be modified by automorphisms of the algebra $\mathcal{A}$.

The point at issue is that the *automorphism group* of the algebra is just the non-commutative version of the *group of diffeomorphisms* of a manifold. For instance, if $\mathcal{A} = C^\infty(M)$ for a compact smooth manifold $M$, and if $\alpha \in \mathrm{Aut}(\mathcal{A})$, then each character $\hat{x}$ of $\mathcal{A}$ is the image under $\alpha$ of a unique character $\hat{y}$ (that is, $\alpha^{-1}(\hat{x})$ is also a character, so it equals $\hat{y}$ for some $y \in M$). Write $\phi(x) := y$; then $\phi$ is a continuous bijection on $M$ satisfying $\alpha(f)(x) = f(\phi^{-1}(x))$, and the chain rule for derivatives shows that $\phi$ is itself smooth and hence is a diffeomorphism of $M$. In fine, $\alpha \leftrightarrow \phi$ is a group isomorphism from $\mathrm{Aut}(C^\infty(M))$ onto $\mathrm{Diff}(M)$.

On a noncommutative $*$-algebra, there are many *inner automorphisms*

$$\sigma_u(a) := u\, a\, u^{-1},$$

where $u$ lies in the unitary group $\mathcal{U}(\mathcal{A})$; these are of course trivial when $\mathcal{A}$ is commutative. We adopt the attitude that these inner automorphisms are henceforth to be regarded as 'internal diffeomorphisms' of our algebra $\mathcal{A}$.

Already in the commutative case, diffeomorphisms change the metric on a manifold. To select a particular metric, some sort of variational principle may be used. In general relativity, one works with the Einstein–Hilbert action

$$I_{\mathrm{EH}} \propto \int_M s(x)\, \sqrt{g(x)}\, d^n x = \int_M s(x)\, \Omega,$$

where $s$ is the scalar curvature of the metric $g$, in order to select a metric minimizing this action. In Yang–Mills theories of particle physics, the bosonic action functional is of the form $I_{\mathrm{YM}} \propto \int F(\star F)$ where $F$ is a gauge field, i.e., a curvature 2-form.

The question then arises as to what is the general prescription for appropriate action functionals in noncommutative geometry.

**Inner automorphisms and gauge potentials.**   Recall how inner automorphisms act on spin geometries: if $u$ is a unitary element of $\mathcal{A}$, the operator $U := uJuJ^\dagger$ implements a unitary equivalence (7.1) between the spin geometries determined by $D$ and by

$$^uD = D + u[D, u^*] \pm Ju[D, u^*]J^\dagger.$$

More generally, any selfadjoint $\mathbb{A} \in \Omega^1_D$ gives rise to a Morita equivalence (7.12) between the geometries determined by $D$ and by $D + \mathbb{A} \pm J\mathbb{A}J^\dagger$.

It is important to observe that these gauge transformations are *trivial* when the geometry is commutative. Recall, from Chapter 3 that this means that $\mathcal{A}$ is commutative and that $\pi^\circ(b) = \pi(b)$: the charge conjugation $J$ on spinors intertwines multiplication by a function with multiplication by its complex conjugate. Therefore we can write $a = Ja^*J^\dagger$ in this case. But then, the first-order property entails

$$Ja[D, b]J^\dagger = a^*J[D, b]J^\dagger = J[D, b]J^\dagger a^*$$
$$= [JDJ^\dagger, JbJ^\dagger]a^* = \pm[D, b^*]a^* = \mp(a[D, b])^*$$

since $JDJ^\dagger = \pm D$. Hence $J\mathbb{A}J^\dagger = \mp\mathbb{A}^*$ for $\mathbb{A} \in \Omega^1_D$, and thus $\mathbb{A} \pm J\mathbb{A}J^\dagger = \mathbb{A} - \mathbb{A}^*$; for a selfadjoint gauge potential, $\mathbb{A} \pm J\mathbb{A}J^\dagger$ vanishes.

As pointed out in [159], this means that, within our postulates, a commutative manifold could support gravity but not electromagnetism; in other words, even to get abelian gauge fields we need that the underlying manifold be noncommutative!

## 8.2  The fermionic action

In the Standard Model of particle physics, the following prescription defines the fermionic action functional:

$$I_F(\xi, \mathbb{A}) := \langle \xi \mid (D + \mathbb{A} \pm J\mathbb{A}J^\dagger)\xi \rangle \tag{8.1}$$

(with the same $\pm$ sign as before). Here $\xi$ may be interpreted as a multiplet of spinors representing elementary particles and antiparticles [37], [159], [194].

The gauge group $\mathcal{U}(\mathcal{A})$ acts on potentials in the following way. If $u \in \mathcal{A}$ is unitary and if $\nabla = \mathrm{Ad}_D + \mathbb{A}$ is a Hermitian connection (7.11), then so also is

$$u\nabla u^* = u\,\mathrm{Ad}_D\,u^* + u\mathbb{A}u^* = \mathrm{Ad}_D + u[D, u^*] + u\mathbb{A}u^*,$$

so that $^u\mathbb{A} := u\mathbb{A}u^* + u[D, u^*]$ is the gauge-transformed potential. With $U = uJuJ^\dagger$, we get $U\mathbb{A}U^{-1} = u\mathbb{A}u^{-1}$ since $JuJ^\dagger$ commutes with $\Omega_D^1$, and so

$$D + {}^u\mathbb{A} \pm J\,{}^u\mathbb{A}J^\dagger = D + u[D, u^*] \pm Ju[D, u^*]J^\dagger + u\mathbb{A}u^* \pm Ju\mathbb{A}u^*J^\dagger$$
$$= U(D + \mathbb{A} \pm J\mathbb{A}J^\dagger)U^{-1}.$$

The gauge invariance of (8.1) under the group $\mathcal{U}(\mathcal{A})$ is now established by

$$I_F(U\xi, {}^u\mathbb{A}) = \langle U\xi | (D + {}^u\mathbb{A} \pm J\,{}^u\mathbb{A}J^\dagger)U\xi \rangle = \langle U\xi | U(D + \mathbb{A} \pm J\mathbb{A}J^\dagger)\xi \rangle = I_F(\xi, \mathbb{A}).$$

**A remark on curvature.** In Yang–Mills models, the fermionic action is supplemented by a *bosonic action* that is a quadratic functional of the gauge fields or curvatures associated to the gauge potential $\mathbb{A}$. One may formulate the curvature of a connection in noncommutative geometry and obtain a Yang–Mills action; indeed, this is the main component of the Connes–Lott models [50]. One can formally introduce the curvature as $\mathbb{F} := d\mathbb{A} + \mathbb{A}^2$, where the notation means

$$d\mathbb{A} := \sum_j [D, a_j][D, b_j] \quad \text{whenever} \quad \mathbb{A} = \sum_j a_j[D, b_j]. \tag{8.2}$$

Regrettably, this definition is flawed, since the first sum may be nonzero in cases where the second sum vanishes [37, VI.1]. For instance, in the commutative case, $a[\not{D}, a] - [\not{D}, \frac{1}{2}a^2] = -i\gamma(a\,da - d(\frac{1}{2}a^2)) = 0$ but $[\not{D}, a][\not{D}, a] = -\gamma(da)^2 = -(da \mid da) < 0$ in general. If we push ahead anyway, we can make a formal check that $\mathbb{F}$ transforms under the gauge group $\mathcal{U}(\mathcal{A})$ by $^u\mathbb{F} = u\mathbb{F}u^*$. Indeed,

$$d(^u\mathbb{A}) = [D, u][D, u^*] + \sum_j [D, ua_j][D, b_ju^*] - \sum_j [D, ua_jb_j][D, u^*]$$
$$= [D, u][D, u^*] + [D, u]\mathbb{A}u^* - u\mathbb{A}[D, u^*] + \sum_j u[D, a_j][D, b_j]u^*,$$

while, using the identity $u[D, u^*]u = -[D, u]$, we get

$$(^u\mathbb{A})^2 = u[D, u^*]u[D, u^*] + u[D, u^*]u\mathbb{A}u^* + u\mathbb{A}[D, u^*] + u\mathbb{A}^2u^*$$
$$= -[D, u][D, u^*] - [D, u]\mathbb{A}u^* + u\mathbb{A}[D, u^*] + u\mathbb{A}^2u^*,$$

and consequently,

$$^u\mathbb{F} := d(^u\mathbb{A}) + (^u\mathbb{A})^2 = u(d\mathbb{A} + \mathbb{A}^2)u^* = u\mathbb{F}u^*.$$

Provided that the definition (8.2) can be corrected, one can then define a gauge-invariant action [100] as the symmetrized Yang–Mills type functional

$$I_B(\mathbb{A}) := \fint (\mathbb{F} + J\mathbb{F}J^\dagger)^2\, ds^n = \fint (\mathbb{F} + J\mathbb{F}J^\dagger)^2\, |D|^{-n},$$

because the noncommutative integral is a trace, so that

$$\fint (^u\mathbb{F} + J\,{}^u\mathbb{F}J^\dagger)^2\, |^uD|^{-n} = \fint U(\mathbb{F} + J\mathbb{F}J^\dagger)^2\, |D|^{-n}U^* = \fint (\mathbb{F} + J\mathbb{F}J^\dagger)^2\, |D|^{-n}.$$

The ambiguity in (8.2) can be removed by introducing the $\mathcal{A}$-bimodule $(\Omega_D^1)^2/J_2$, where the subbimodule $J_2$ consists of the so-called 'junk' terms $\sum_j [D, a_j][D, b_j]$ for which $\sum_j a_j[D, b_j] = 0$. Then, by redefining $\mathbb{F}$ as the orthogonal projection of $d\mathbb{A} + \mathbb{A}^2$ on the orthogonal complement of $J_2$ in $(\Omega_D^1)^2$, one gets a well-defined curvature and the noncommutative integral of its square gives the desired Yang–Mills action.

## 8.3 The spectral action principle

This Yang–Mills action, evaluated on a suitable spin geometry, achieves the remarkable feat of reproducing the classical Lagrangian of the Standard Model. This is discussed at length in [37, VI] and in several other places [18], [24], [118], [130], [159], [192]. However, its computation leads to fearsome algebraic manipulations and very delicate handling of the junk terms, leading one to question whether this action is really fundamental.

The seminal paper [40] makes an alternative proposal. The unitary equivalence $D \mapsto D + u[D, u^*] \pm Ju[D, u^*]J^\dagger$ is a perturbation by internal diffeomorphisms, and one can regard the Morita equivalence $D \mapsto D + \mathbb{A} \pm J\mathbb{A}J^\dagger$ as an *internal fluctuation* of $D$. The correct bosonic action functional should not merely be *diffeomorphism invariant* – where by diffeomorphisms we mean automorphisms of $\mathcal{A}$ – that is to say, 'of purely gravitational nature', but one can go further and ask that it be *spectrally invariant*. As stated unambiguously by Chamseddine and Connes [26]:

"The physical action only depends upon $\mathrm{sp}(D)$."

The fruitfulness of this viewpoint has been exemplified by Landi and Rovelli [141], [142], [211], who consider the eigenvalues of the Dirac operator as dynamical variables for general relativity.

Since quantum corrections must still be provided for [2], [3], the particular action chosen should incorporate a cutoff scale $\Lambda$ (roughly comparable to inverse Planck length, or Planck mass, where the commutative spacetime geometry must surely break down), and some suitable cutoff function: $\phi(t) \geq 0$ for $t \geq 0$ with $\phi(t) = 0$ for $t \gg 1$. Therefore, Chamseddine and Connes proposed a bosonic action of the form

$$B_\phi(D) = \mathrm{Tr}\, \phi(D^2/\Lambda^2). \tag{8.3}$$

This spectral action turns out to include not only the Standard Model bosonic action but also the Einstein–Hilbert action for gravity, plus some higher-order gravitational terms, thereby establishing it firmly as an action for an effective field theory at low energies. We refer to [26], [116], see also [27], for the details of how all these terms emerge in the calculation. Most of them can also be recovered by an alternative procedure involving Quillen's superconnections [85], which seems to suggest that the Chamseddine–Connes action is in the nature of things.

Here we limit ourselves to the computational task of explaining the general method of extracting such terms from (8.3), by a spectral asymptotic development in the cutoff parameter $\Lambda$.

## 8.4 Spectral densities and asymptotics

We consider the general problem of providing the functional (8.3) with an asymptotic expansion as $\Lambda \to \infty$, without prejudging the particular cutoff function $\phi$. In any case, as we shall see, the dependence of the final results on $\phi$ is very weak. There is, of course, a great deal of accumulated experience with the related heat kernel expansion for pseudodifferential operators [6], [96], [217]. One can adapt the heat kernel expansion to develop (8.3), under the tacit assumption that $\phi$ is a Laplace transform [26]. However, we take a more direct route, avoiding the detour through the heat kernel expansion.

The basic idea, expounded in detail in [80], see also [81] and [104, Sec. 7.4], is to develop *distributional* asymptotics directly from the *spectral density* of the positive selfadjoint operator $A = D^2$. If the spectral projectors of $A$ are $\{E(\lambda) : \lambda \geq 0\}$, the spectral density is the derivative

$$\delta(\lambda - A) := \frac{dE(\lambda)}{d\lambda},$$

which makes sense as a distribution with operatorial values in $\mathcal{L}(\mathcal{H}^\infty, \mathcal{H})$. For instance, when $A$ has discrete spectrum $\{\lambda_j\}$ (in increasing order) with a corresponding orthonormal basis of eigenfunctions $u_j$, then

$$E(\lambda) = \sum_{\lambda_j \leq \lambda} |u_j\rangle\langle u_j|, \quad \text{and so} \quad d_A(\lambda) = \sum_{j=1}^{\infty} |u_j\rangle\langle u_j| \, \delta(\lambda - \lambda_j).$$

A functional calculus may be defined by setting

$$f(A) := \int_0^\infty f(\lambda) \, \delta(\lambda - A) \, d\lambda.$$

For further details of this calculus and the conditions for its validity, we refer to [79], [80], [104].

**Spectral densities of pseudodifferential operators.** The algebra of the Standard Model spectral triple is of the form $C^\infty(M) \otimes \mathcal{A}_F$, where $M$ is a compactified (Euclidean) spacetime and $\mathcal{A}_F$ is an algebra with a finite basis; in fact, $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, acting on a spinor multiplet space $L^2(M, S) \otimes \mathcal{H}_F \simeq L^2(M, S \otimes \mathcal{H}_F)$ through a finite-dimensional real representation [192]. This particular finite-dimensional algebra

is selected almost uniquely by the physical properties of the known particle spectrum [119], [136], [193]. The operator $D$ is of the form $\not{D} \otimes 1 + \gamma_5 \otimes D_F$, where $D_F$ is a matrix of Yukawa mass terms and $\not{D}$ is the Dirac operator on the spinor space of $M$. The Lichnerowicz formula (2.10) gives $\not{D}^2 = \Delta^S + \frac{1}{4}s$, where $s$ is the scalar curvature. After incorporating the terms from $D_F$, one finds [26], [116] that $D^2$ is a generalized Laplacian [12] with matrix-valued coefficients. Thus the task is to compute an expansion for (8.3) under the assumption that $A = D^2$ is a pseudodifferential operator of order 2.

Suppose, then, that $A$ is a positive, elliptic, (scalar) classical pseudodifferential operator of order $d$ on an $n$-dimensional manifold $M$. If $A$ has the symbol $\sigma(A) = a(x, \xi)$ in local coordinates, we ask what the symbol $\sigma(\delta(\lambda - A))$ might be. When $A$ has constant coefficients, the symbol of $A^k$ is just

$$a(x, \xi)^k = \int_0^\infty \lambda^k \, \delta(\lambda - a(x, \xi)) \, d\lambda,$$

so $\delta(\lambda - a(x, \xi))$ is the symbol of $\delta(\lambda - A)$ in that particular case. In general, the symbol of $A^k$ depends also on the derivatives of $a(x, \xi)$, so we arrive at the prescription [80]:

$$\sigma\big(\delta(\lambda - A)\big) \sim \delta(\lambda - \sigma(A)) - q_1 \, \delta'(\lambda - \sigma(A)) + q_2 \, \delta''(\lambda - \sigma(A)) \\ - \cdots + (-1)^k q_k \, \delta^{(k)}(\lambda - \sigma(A)) + \cdots \quad \text{as } \lambda \to \infty. \tag{8.4}$$

By computing $\int \lambda^k \, \sigma(\delta(\lambda - A)) \, d\lambda$ for $k = 0, 1, 2, \ldots$, we get $q_1 = 0$, and

$$q_2(x, \xi) = \tfrac{1}{2}\big(\sigma(A^2) - \sigma(A)^2\big), \quad q_3(x, \xi) = \tfrac{1}{6}\big(\sigma(A^3) - 3\sigma(A^2)\sigma(A) + 2\sigma(A)^3\big),$$

and so on. The order of the symbol $q_2$ is $\leq (2d - 1)$, the order of $q_3$ is $\leq (3d - 2)$.

**Cesàro calculus.** An important technical issue is how to interpret the distributional development (8.4). On subtracting the first $N$ terms on the right from the left hand side, one needs a distribution that falls off like $\lambda^{\alpha_N}$ as $\lambda \to \infty$, with exponents $\alpha_N$ that decrease to $-\infty$. It turns out that this holds, *in a Cesàro-averaged sense* [78], [80], [81]. A distribution $f$ is said to be of order $\lambda^\alpha$ at infinity, in the Cesàro sense, and we write

$$f(\lambda) = O(\lambda^\alpha) \quad \text{(C)} \quad \text{as } \lambda \to \infty,$$

if for some $N$, there is a function $f_N$ whose $N$th derivative is $f$ and a polynomial $p_N$ of degree $< N$, such that $f_N(\lambda) = p_N(\lambda) + O(\lambda^{\alpha+N})$ as $\lambda \to \infty$.

If $\sum_{n=1}^\infty a_n$ is a Cesàro-summable series, then $f(\lambda) := \sum_{n=1}^\infty a_n \delta(\lambda - n)$ is a distribution satisfying

$$\int_0^\infty f(\lambda) \, d\lambda \sim \sum_{n=1}^\infty a_n \quad \text{(C)}.$$

If $A$ has symbol $a(x, \xi)$ by (5.7), then $Au(x) = \int k_A(x, y)u(y)\, d^n y$, where the integral kernel is the distribution given in local coordinates by

$$k_A(x, y) := (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, \xi)\, d^n \xi.$$

In particular, on the diagonal,

$$k_A(x, x) = (2\pi)^{-n} \int a(x, \xi)\, d^n \xi.$$

The kernel for the spectral density $\delta(\lambda - A)$ is then given on the diagonal by

$$d_A(x, x; \lambda) \sim (2\pi)^{-n} \int \left[ \delta(\lambda - a(x, \xi)) + q_2(x, \xi)\, \delta''(\lambda - a(x, \xi)) - \cdots \right] d^n \xi \quad \text{(C)}.$$
$$(8.5)$$

By functional calculus, the action functional (8.3) may then be expressed as

$$\operatorname{Tr} \phi(D^2/\Lambda^2) = \int_M \int_0^\infty \phi(\lambda/\Lambda^2) \operatorname{tr} d_{D^2}(x, x; \lambda)\, d\lambda\, \sqrt{g(x)}\, d^n x,$$

provided one learns the trick of integrating a Cesàro development to get a parametric development in $t = \Lambda^{-2}$ as $t \downarrow 0$.

**Parametric developments.** Some distributions have zero Cesàro expansion, namely those $f$ for which $f(\lambda) = o(\lambda^{-\infty})$ (C) as $|\lambda| \to \infty$. These coincide with the dual space $\mathcal{K}'$ of the space $\mathcal{K}$ of GLS symbols [106]: elements of $\mathcal{K}$ are smooth functions $\phi$ such that for some $\alpha$, $\phi^{(k)}(\lambda) = O(|\lambda|^{\alpha-k})$ as $|\lambda| \to \infty$. The space $\mathcal{K}$ includes all polynomials, so any $f \in \mathcal{K}'$ has *moments* $\mu_k := \int \lambda^k f(\lambda)\, d\lambda$ of all orders. Indeed, $\mathcal{K}'$ is precisely the space of distributions that satisfy the *moment asymptotic expansion* [81]:

$$f(\sigma\lambda) \sim \sum_{k=0}^\infty \frac{(-1)^k \mu_k\, \delta^{(k)}(\lambda)}{k!\, \sigma^{k+1}} \quad \text{as } \sigma \to \infty.$$

This a parametric development of $f(\lambda)$. A more general distribution may have a Cesàro expansion in falling powers of $\lambda$:

$$f(\lambda) \sim \sum_{k \geq 1} c_k \lambda^{\alpha_k} \quad \text{(C)} \quad \text{as } \lambda \to \infty,$$

and the corresponding parametric development is of the form [81]:

$$f(\sigma\lambda) \sim \sum_{k \geq 1} c_k (\sigma\lambda)^{\alpha_k} + \sum_{m \geq 0} \frac{(-1)^m \mu_m\, \delta^{(m)}(\lambda)}{m!\, \sigma^{m+1}} \quad \text{as } \sigma \to \infty. \quad (8.6)$$

(This is an oversimplification, valid only if no $\alpha_k$ is a negative integer: the general case is treated in [81] and utilized in [80].)

The moral is this: if one knows the Cesàro development, the parametric development is available also, assuming that one can compute the moments that appear in (8.6). Then one can evaluate on a test function by a change of variable, obtaining an ordinary asymptotic expansion in a new parameter:

$$\int f(\lambda)\phi(t\lambda)\,d\lambda \sim \sum_{k\geq 1} c_k\, t^{-\alpha_k-1} \int_0^\infty \lambda^{\alpha_k}\phi(\lambda)\,d\lambda + \sum_{m\geq 0} \frac{\mu_m\,\phi^{(m)}(0)}{m!}\, t^m \quad (8.7)$$

as $t \downarrow 0$. The integral on the right should be regarded as a finite-part integral; also, when some $\alpha_k$ are negative integers, there are extra terms in $t^r \log t$. The heat kernel expansion may be obtained by taking $\phi(\lambda) := e^{-\lambda}$ for $\lambda \geq 0$.

**The spectral coefficients.** The coefficients of the spectral density kernel (8.5), after integration over $\xi$, have an intrinsic meaning: they are all Wodzicki residues! It has been argued in [80] that (8.5) simplifies to

$$d_A(x, x; \lambda)\, d^n x \sim \frac{1}{d\,(2\pi)^n}\Big[\mathrm{wres}_x(A^{-n/d})\lambda^{(n-d)/d}$$
$$+ \mathrm{wres}_x(A^{(1-n)/d})\lambda^{(n-d-1)/d} \quad\quad\quad\quad (8.8)$$
$$+ \cdots + \mathrm{wres}_x(A^{(k-n)/d})\lambda^{(n-d-k)/d} + \cdots\Big] \; (C) \quad \text{as } \lambda \to \infty.$$

It is worth indicating briefly how this comes about: we shall compute the leading term in (8.8). To integrate (8.5), we use polar coordinates $\xi = r\omega$ with $|\omega| = 1$. Since the integrand involves $\delta(\lambda - a(x, r\omega))$ and its derivatives, we must solve $a(x, r\omega) = \lambda$ for $r$; denote the unique positive solution by $r = r(x, \omega; \lambda)$. To solve, we revert the expansion

$$\lambda = a(x, r\omega) \sim \sum_{j\geq 0} a_{d-j}(x, \omega)\, r^{d-j} \quad \text{as } r \to \infty \quad\quad (8.9)$$

to get a development in falling powers of $\lambda$:

$$r = r(x, \omega; \lambda) \sim \sum_{k\geq 0} r_k(x, \omega)\, \lambda^{(1-k)/d} \quad \text{as } \lambda \to \infty. \quad\quad (8.10)$$

Now we unpack the distribution

$$\delta(\lambda - a(x, r\omega)) = \frac{\delta(r(x, \omega; \lambda))}{a'(x, r(x, \omega; \lambda)\omega)}.$$

Since $d^n\xi = \sigma_\omega\, r^{n-1}\, dr$, the first term in (8.5) yields

$$(2\pi)^{-n} \int_{|\omega|=1} \frac{r^{n-1}(x, \omega; \lambda)}{a'(x, r(x, \omega; \lambda)\omega)}\, \sigma_\omega. \quad\quad (8.11)$$

If we retain only the first terms in (8.9) and (8.10), this integrand becomes

$$\frac{r^{n-1}}{d\, r^{d-1} a_d(x, \omega)} \sim \frac{r_0(x, \omega)^{n-d} \lambda^{(n-d)/d}}{d\, a_d(x, \omega)} \sim \frac{\lambda^{(n-d)/d}}{d} a_d(x, \omega)^{-n/d},$$

since $r_0(x, \omega) = a_d(x, \omega)^{-1/d}$. Thus the leading term in the $\lambda$-development of (8.11) is

$$\frac{\lambda^{(n-d)/d}}{d\, (2\pi)^n} \int_{|\omega|=1} a_d(x, \omega)^{-n/d}\, \sigma_\omega.$$

It remains only to notice that $a_d(x, \omega)^{-n/d}$ is the principal symbol, of order $(-n)$, of the operator $A^{-n/d}$, and to apply (5.11).

**Spectral densities of generalized Laplacians.** We can apply this general machinery to the case where $A$ is a generalized Laplacian, with a symbol of the form

$$a(x, \xi) = -g^{ij}(x)\, \xi_i \xi_j + b^k(x) \xi_k + c(x),$$

where $b^k$, $c$ are scalar functions. Rewriting (8.8) as

$$d\, (2\pi)^n\, d_A(x, x; \lambda) \sim a_0(x) \lambda^{(n-d)/d} + a_1(x) \lambda^{(n-d-1)/d} + \cdots,$$

we see that $a_0(x)$ is constant with value $\Omega_n$. Also, $a_k = 0$ for odd $k$ since their computation involves integrating odd powers of the $\omega_j$ over the sphere $|\omega| = 1$. For $a_2(x)$, one can express the metric in normal coordinates [12] near $x = x_0$:

$$g_{ij}(x) \sim \delta_{ij} + \tfrac{1}{3} R_{iklj}(x_0)\, (x - x_0)^k (x - x_0)^l + \cdots$$

where $R_{iklj}$ is the Riemann curvature tensor. The $q_2$ term of (8.5) extracts from this a Ricci-tensor term $\tfrac{1}{3} R_{kj}(x)\, \xi_k \xi_j$ and integration over the unit sphere leaves the scalar curvature $s(x)$. The upshot is that

$$a_2(x) = \frac{(n-2)\Omega_n}{2} \left(\tfrac{1}{6} s(x) - c(x)\right). \qquad (8.12)$$

On the other hand, this gives the Wresidue density of $A^{(2-n)/2}$. We thus arrive at one of the most striking results in noncommutative geometry, derived independently by Kastler [128] and Kalau and Walze [122], [219], namely that the Einstein–Hilbert action of general relativity is a multiple of the Wodzicki residue of $D\!\!\!/^{-2}$ on a 4-dimensional manifold:

$$\text{Wres } D\!\!\!/^{-2} \propto \int_M s(x)\, \sqrt{g(x)}\, d^4x,$$

on combining (8.12) with the Lichnerowicz formula $c = \tfrac{1}{4} s$. The computation of Wres $D^{-2}$ for the Standard Model is given in [26], [116].

**The Chamseddine–Connes action.**  Pulling all the threads together, we apply the expansion (8.8) to the action functional (8.3). For the Standard Model plus gravity, we take $n = 4$ and $A = D^2$, a generalized Laplacian, acting now on a space of sections of the spinor bundle $S \to M$. From (8.8) we get just two terms:

$$\operatorname{tr} d_{D^2}(x, x; \lambda) \sim \frac{4}{16\pi^2}\lambda + \frac{1}{32\pi^4}\operatorname{wres}_x D^{-2} \quad (C) \quad \text{as } \lambda \to \infty$$

since the nonnegative powers of the differential operator $D^2$ have zero Wresidue. Applying (8.7) with $t = \Lambda^{-2}$ gives an expansion of the form

$$\operatorname{Tr} \phi(D^2/\Lambda^2) \sim \frac{1}{4\pi^2}\phi_0 \Lambda^4 + \frac{1}{32\pi^4}\operatorname{Wres} D^{-2}\phi_2 \Lambda^2 + \sum_{m \geq 0} b_{2m+4}(D^2)\phi_{2m+4} \Lambda^{-2m}$$

as $\Lambda \to \infty$, where

$$\phi_0 = \int_0^\infty \lambda\phi(\lambda)\, d\lambda; \quad \phi_2 = \int_0^\infty \phi(\lambda)\, d\lambda; \quad \phi_{2m+4} = (-1)^m \phi^{(m)}(0), \ m \in \mathbb{N}.$$

Thus the cutoff function $\phi$ plays only a minor role, and these integrals and derivatives may be determined from experimental data.

In [26] detailed results are given for the spectral triple associated to the Standard Model. The bosonic parts of the SM appear in the $\Lambda^2$ and $\Lambda^0$ terms; the Einstein–Hilbert action appears in the $\Lambda^2$ term, as expected; and other gravity pieces and a gravity-Higgs coupling in the $\Lambda^0$ term; the $\Lambda^4$ term is cosmological. The $\Lambda^0$ term is conformally invariant. The higher-order terms may safely be neglected.

More recently [27], it has been shown that this spectral action can be extended by coupling to a dilaton field, in order to compensate for the lack of scale invariance of the Higgs sector mass terms. How robust the spectral action may be under more general conformal transformations remains to be seen.

# Epilogue: new directions

The panorama of noncommutative geometry laid out in the preceding chapters corresponds to a snapshot taken some years ago. Much has happened in the intervening years, and here we give a brief *aggiornamento* taking stock of a few of the more recent developments.

## 9.1 Noncommutative field theories

In a groundbreaking paper, Connes, Douglas and Schwarz [45] showed that compactification of M-theory, in the context of dimensionally reduced gauge theory actions, leads to spaces with embedded noncommutative tori. Soon after, Douglas and Hull saw that gauge theories on noncommutative spaces arise naturally from string theory [69]. For more on the relation between noncommutative geometry and strings, see [139] and [154].

Seiberg and Witten [198] considered open strings with allowed endpoints on 2D-branes in a B-field background, and found that the endpoints live on a noncommutative space whose spacetime coordinates must satisfy commutation relations of the form

$$[x^\mu, x^\nu] = i\,\theta^{\mu\nu}, \tag{9.1}$$

for some skewsymmetric matrix $\theta = [\theta^{\mu\nu}]$ of real scalars. This revived an old proposal by Snyder [201] for a quantization of spacetime, that had also been reincarnated by Doplicher *et al.* [68]: for its prehistory, see [120]. Now, (9.1) is just an infinitesimal form of the basic relation (4.3) defining a noncommutative torus (of dimension four), and the $x^\mu$ appear as generators of the algebra of 'noncommutative $\mathbb{R}^4$'. But the product in this algebra is none other than the Moyal product (6.5), where $\theta$ replaces $\hbar\,1_4$ and $\mathbb{R}^4$ is no longer interpreted as a phase space.

This startling appearance of the Moyal product in string theories had been foreshadowed a few years earlier by Filk [86], who showed that one can adapt the formalism of a scalar Lagrangian field theory to the Moyal product algebra. For instance, in $\phi_4^4$ theory, the action functional becomes

$$S = \int \left( \frac{1}{2} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x_\mu} + \frac{m^2}{2} \phi^2 + \frac{g}{4!} \phi \star_\theta \phi \star_\theta \phi \star_\theta \phi \right) d^4x.$$

This deformation of the usual $\phi_4^4$ theory contains a nonlocal interaction; in the perturbative expansion of the Green function, new terms appear, labelled by 'nonplanar'

Feynman graphs, but they never cancel the ultraviolet divergences due to the 'planar' graphs of the undeformed theory. Thus, giving up locality does not yield a better ultraviolet behaviour of the field theory. The same point was made in [216], in a model-independent context: with any noncommutative one-particle space described by a spectral triple, the overall ultraviolet behaviour remains the same as that of an ordinary space of the same dimension. Quantum field theories on noncommutative spaces still require renormalization.

The topological difficulties that arise from dealing with nonplanar graphs were completely overcome in two remarkable papers by Chepelev and Roiban [28], [29], who determined the precise conditions under which nonplanar graphs yield convergent integrals and showed how to adapt the recursive subtraction of counterterms to the noncommutative case.

A more serious difficulty with the spacetime relations was pointed out by Gomis and Mehen [98]: any timelike noncommutativity $\theta^{0i} \neq 0$ leads to a violation of unitarity of the $S$-matrix. One may still proceed with purely spacelike noncommutativity parameters, but even so, other troubles arise: for instance, the one-loop effective action may not exist [92]. A more optimistic perspective on deformed field theories is offered in [224].

The literature on noncommutative field theories is already vast. A good source book, up to 2002, is [153]. For the overall picture, the excellent reviews [70] and [206] are recommended.

## 9.2 Isospectral deformations

Since the year 2000, several new examples of spectral triples, many of which support spin geometries, have emerged. Chief among them are the isospectral deformations of commutative spectral triples, pioneered by Connes and Landi [49]. Instead of fixing the algebra of a given spectral triple (or its Morita equivalence class) and deforming or 'fluctuating' the metric-generating operator $D$, as we have seen in Chapter 7, the new procedure consists of fixing $D$ on a given Hilbert space $\mathcal{H}$ and deforming the algebra $\mathcal{A}$ to a new algebra that is also represented on $\mathcal{H}$.

A problem posed in [42] was: can one recover the metric geometry of the Riemann sphere $\mathbb{S}^2$ from spectral triple data alone? Thus, if $\mathcal{A}$ is a dense $*$-subalgebra of some $C^*$-algebra $A$ containing elements $x, y, z$, and if the matrix

$$p = \frac{1}{2} \begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix} \in M_2(\mathcal{A})$$

is a projector, i.e., $p = p^* = p^2$ in $M_2(\mathcal{A})$, it is easy to show that the projector relations imply that $x, y, z$ are commuting selfadjoint elements of $\mathcal{A}$, and that $x^2 + y^2 + z^2 = 1$. Thus, $A = C(X)$, where $X \subseteq \mathbb{S}^2$ is a closed subset. If $\mathcal{A}$ is the algebra of some spin

geometry $(\mathcal{A}, \mathcal{H}, D; \Gamma, J)$, and if $\boldsymbol{c} := \mathrm{tr}\left(\left(p - \frac{1}{2}\right) \otimes p \otimes p\right)$, then $\boldsymbol{c}$ is a Hochschild 2-cycle over $\mathcal{A}$ and the condition $\pi_D(\boldsymbol{c}) = \Gamma$ can only hold if $X$ is the support of the area measure on the sphere, so that $X = \mathbb{S}^2$.

In the same way, one can obtain the sphere $\mathbb{S}^4$ with its round metric by starting with an analogous projector in $M_4(\mathcal{A})$:

$$p = \begin{pmatrix} (1+z)1_2 & q \\ q^* & (1-z)1_2 \end{pmatrix}, \quad \text{where } q = \begin{pmatrix} a & b \\ -b^* & a \end{pmatrix}, \qquad (9.2)$$

and imposing conditions that guarantee that $\boldsymbol{c} := \mathrm{tr}\left(\left(p - \frac{1}{2}\right) \otimes p \otimes p \otimes p \otimes p\right)$ is a Hochschild 4-cycle over the $*$-algebra $\mathcal{A}$ generated by $a, b, z$. Then $\mathcal{A}$ is commutative, $z^* = z$, and the 4-sphere relation $a^*a + b^*b + z^2 = 1$ holds. The orientation condition $\pi_D(\boldsymbol{c}) = \Gamma$ entails that $M(\mathcal{A}) = \mathbb{S}^4$. But there is now a noncommutative solution, if one replaces the entry $-b^*$ in (9.2) by $-\lambda b^*$, where $\lambda = e^{2\pi i\theta} \in \mathbb{T}$. The resulting (pre-$C^*$-)algebra is somewhat whimsically called $\mathcal{A} = C^\infty(\mathbb{S}^4_\theta)$ where the noncommutative space $\mathbb{S}^4_\theta$ is thought of as a 'deformed 4-sphere'. This algebra is represented on the spinor space $\mathcal{H}$ of $\mathbb{S}^4$ and, using the Dirac operator $\slashed{D}$ for the round metric on $\mathbb{S}^4$, we get a deformed spin geometry $(C^\infty(\mathbb{S}^4_\theta), \mathcal{H}, \slashed{D}; \Gamma, J)$.

This example can be generalized in two ways. One can obtain 'noncommutative spheres' in other dimensions by imposing conditions on suitable matrices of generators (in odd dimensions, these matrices should be unitary, since they must represent an odd $K$-theory class) that yield a Hochschild cycle of top degree, and classify the resulting algebras. For 3-spheres, this program has been successfully carried by Connes and Dubois-Violette [46], [47]; it yields a three-parameter family of noncommutative 3-spheres. These include a one-parameter subfamily $\mathbb{S}^3_\theta$ of the Connes–Landi type. The other 3-spheres have deep and intricate relations with the theory of elliptic curves [47].

The other, simpler, generalization is to '$\theta$-deform' any compact Riemannian manifold $M$ that admits an action of the torus $\mathbb{T}^l$, for $l \geq 2$. The induced action of $\mathbb{T}^l$ on the algebra $C^\infty(M)$ can be deformed to a noncommutative algebra $C^\infty(M_\theta)$, by replacing the ordinary product of functions by a *Moyal* product. Indeed, if $A$ is any $C^*$-algebra with a strongly continuous action $\alpha$ of $\mathbb{R}^l$, and $\mathcal{A}$ is the smooth domain of this action, then $\mathcal{A}$ is a Fréchet pre-$C^*$-algebra dense in $A$; and if $\theta \in M_l(\mathbb{R})$ is skewsymmetric, then the $\theta$-deformed product [183] is defined on $\mathcal{A}$ by

$$a \star_\theta b := \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \alpha_{\frac{1}{2}\theta u}(a) \, \alpha_{-t}(b) \, e^{2\pi i u t} \, d^l u \, d^l t.$$

This generalizes (6.5), after a change of variables [91].

In particular, if the action $\alpha$ is *periodic*, that is, $\alpha_{t+r} = \alpha_t$ for $r \in \mathbb{Z}^l$, so that $\alpha$ is effectively an action of $\mathbb{T}^l$, then the deformed algebra $\mathcal{A}_\theta$ decomposes into spectral subspaces for this torus action, and it can be described by generalized Fourier series. For instance, the pre-$C^*$-algebra $\mathcal{A} = C^\infty(\mathbb{S}^4)$, with generators $a, b, z$, carries an action of $\mathbb{T}^2$ given by $(t_1, t_2) \triangleright (a, b, z) := (t_1 a, t_2 b, z)$; the resulting Moyal deformation is $\mathcal{A}_\theta = C^\infty(\mathbb{S}^4_\theta)$.

When the manifold $M$ is spin, the action of $\mathbb{T}^l$ can be lifted to a torus action on the spinor bundle. If $M$ is gifted with a $\mathbb{T}^l$-invariant metric, then the Dirac operator $D\!\!\!\!/$, the charge conjugation $J$ and the grading $\Gamma$ (if $\dim M$ is even) are invariant under the lifted action, and $(C^\infty(M_\theta), \mathcal{H}, D\!\!\!\!/; \Gamma, J)$ is a noncommutative spin geometry that is isospectral to the undeformed case, because $D\!\!\!\!/$ is unchanged [49].

**Quantum symmetry of deformations.** If $M = G/K$ is a homogeneous space of a compact Lie group $G$, where the Lie subgroup $K$ includes a maximal torus $\mathbb{T}^l$ of $G$ with $l \geq 1$, then $C^\infty(G)$ can be deformed by $\Theta = \theta \oplus (-\theta)$ in $M_{2l}(\mathbb{R})$ [184] and then $C^\infty(M_\theta)$ is an embedded homogeneous space for the *compact quantum group* $C^\infty(G_\Theta)$, as was shown in [212]; see also [199].

This means, in more detail, that the Gelfand cofunctor transposes the *left action* of the group $G$ on the quotient space $G/K$ to a *right coaction* of $C^\infty(G)$ on $C^\infty(G/K)$. On a compact Lie group $G$, one can consider three function algebras: the $C^*$-algebra $C(G)$ of continuous functions, the pre-$C^*$-algebra $C^\infty(G)$ of smooth functions, and the 'polynomial' algebra $\mathcal{O}(G)$ of 'representative functions', that is, the $*$-subalgebra of $C^\infty(G)$ generated by the matrix elements of finite-dimensional unitary representations. The algebra $\mathcal{O}(G)$ is a *Hopf algebra*, because it has a *coproduct* $\Delta \colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ – algebraic tensor product – given by $\Delta f(x, y) := f(xy)$, for $x, y \in G$ and 'counit' $\varepsilon(f) := f(1)$; these operations are algebra homomorphisms, and $Sf(x) := f(x^{-1})$ defines a 'coinverse' or 'antipode' on $\mathcal{O}(G)$. These operations extend to smooth and continuous functions, although completed tensor products are needed to extend $\Delta$.

In an arbitrary Hopf algebra $H$, the coproduct $\Delta \colon H \to H \otimes H$ is written $\Delta a =: \sum a_{(1)} \otimes a_{(2)}$ (finite sum) and it is coassociative, that is, $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ as maps from $H$ to $H \otimes H \otimes H$ [125]. A subalgebra $F \subset H$ is an 'embedded homogeneous space' if $\Delta(F) \subset F \otimes H$: the restriction of $\Delta$ to $F$ is the right coaction $F \to F \otimes H$. In the compact Lie group case, we can identify elements of $\mathcal{O}(G/K)$ with functions in $\mathcal{O}(G)$ invariant under right translations from $K$. It turns out [184] that after the above deformations of the function algebras on $G$ or $G/K$, the coproduct is still an algebra homomorphism. The resulting coaction can be regarded as a 'quantum symmetry' on a noncommutative space.

**Noncompact isospectral deformations.** These isospectral deformations have yielded an opportunity to address a vexing problem, namely, how to extend the machinery of spectral triples and spin geometries to *noncompact* noncommutative spaces. By this we mean spin geometries with *nonunital* coordinate algebras. We have noted in Chapter 3 that the bare definition of spectral triple extends to the nonunital case, by demanding compactness of $a(D - \lambda)^{-1}$ for each $a \in \mathcal{A}$ and $\lambda \notin \operatorname{sp} D$. The seven spin geometry conditions are harder to satisfy; but a few of them become moot if we consider an isospectral deformation of the Dirac operator on a noncompact Riemannian manifold.

The simplest case of that is the 'plane' $\mathbb{R}^{2m}$, that we can deform to a *Moyal plane*, by using the Moyal product $\star_\hbar$ of (6.5) based on $\mathbb{R}^{2m} \simeq T^*\mathbb{R}^m$. The issue is which of the

many possible versions of the Moyal *algebra* to use. The Schwartz space $\mathcal{S}(\mathbb{R}^{2m})$ is an algebra under $\star_\hbar$ [102], in fact a pre-$C^*$-algebra [91], but it is too small. The finiteness and orientation properties require that our algebra $\mathcal{A}_\hbar$ have an auxiliary *unitization* $\widetilde{\mathcal{A}}_\hbar$, that is, a unital algebra in which $\mathcal{A}_\hbar$ is embedded, not necessarily densely, as an essential ideal. It turns out that we can take $\widetilde{\mathcal{A}}_\hbar = (\mathcal{B}(\mathbb{R}^{2m}), \star_\hbar)$, where $\mathcal{B}(\mathbb{R}^{2m})$ is the vector space of smooth functions $f$ on $\mathbb{R}^{2m}$, all of whose derivatives $\partial^\alpha f$ are bounded [195]. The appropriate nonunital algebra is then $\mathcal{A}_\hbar = (\mathcal{D}_{L^2}(\mathbb{R}^{2m}), \star_\hbar)$, where $\mathcal{D}_{L^2}(\mathbb{R}^{2m})$ is the vector space of smooth functions, all of whose derivatives are square-integrable [195]. The smooth domain of the Dirac operator $\slashed{D} = -i\,\gamma^\mu\,\partial_\mu$ is then $\mathcal{H}^\infty = (\mathcal{A}_\hbar)^N$ with $N = 2^{\lfloor n/2 \rfloor}$. This is the pullback to $\mathcal{A}_\hbar$ of the finitely generated projective (indeed, free) module $(\widetilde{\mathcal{A}}_\hbar)^N$ over the unital algebra $\widetilde{\mathcal{A}}_\hbar$.

The unital algebra $\widetilde{\mathcal{A}}_\hbar$ also includes the *plane waves* $u_k(x) := e^{2\pi i k \cdot x}$, for $k \in \mathbb{R}^{2m}$, which satisfy
$$u_k \star_\hbar u_l = e^{-\pi i \hbar\, s(k,l)}\, u_{k+l}, \quad \text{for all } k, l \in \mathbb{R}^{2m}.$$
In particular, by taking $k, l \in \mathbb{Z}^{2m}$ to be *integral* vectors, we get a copy of the algebra of a noncommutative $n$-torus $C^\infty(\mathbb{T}_\hbar^n)$ included in $\widetilde{\mathcal{A}}_\hbar$ as a subalgebra of *periodic* functions. The Hochschild $n$-cycle $\boldsymbol{c}$ for this $n$-torus is an analogue of (4.11) with $2m$ generators $v_j = u_{e_j}$, $j = 1, \ldots, 2m$:

$$\boldsymbol{c} = \frac{(2\pi i)^{-2m}}{(2m)!} \sum_{\sigma \in S_{2m}} (-1)^\sigma\, (v_{\sigma(1)} v_{\sigma(2)} \ldots v_{\sigma(2m)})^{-1} \otimes v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(2m)}.$$

In the commutative case, we can write $v_j = e^{2\pi i t_j}$, and the right hand side reduces to $dt_1 \wedge \cdots \wedge dt_n$, the usual volume form for either $\mathbb{R}^n$ or the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Thus the orientation cycle $\boldsymbol{c}$ is a Hochschild cycle over the unitization $\widetilde{\mathcal{A}}_\hbar$.

The metric dimension of the Moyal plane is $2m$, as expected. To prove it, we should be able to check that $a(1 + |\slashed{D}|)^{-2m}$ lies in the Dixmier ideal $\mathcal{L}^{1+}$ for all $a \in \mathcal{A}_\hbar$; but it turns out [91] that this can only be verified for $a$ in the smaller, but still dense, subalgebra $\mathcal{S}(\mathbb{R}^{2m})$. Happily, with this proviso, we can compute its Dixmier trace and get the expected result, compare (5.14):

$$\fint a(1 + |\slashed{D}|)^{-2m} = \frac{1}{m!\,(2\pi)^m} \int_{\mathbb{R}^{2m}} a(x)\, d^{2m}x.$$

This example can be generalized to a large family of noncompact Riemannian spin manifolds (with 'bounded geometry') that admit either an action of the torus $\mathbb{T}^l$, or even a free action of $\mathbb{R}^l$, for $l \geq 2$. The latter case is that of a trivial vector bundle of rank $l$, with $\mathbb{R}^l$ acting by vertical translations. This is done in [93]. The upshot is that the full apparatus of noncommutative spin geometry is available for noncompact isospectral deformations.

## 9.3 Geometries with quantum group symmetry

For a long time, it was thought that well-behaved spectral triples could not be constructed over the coordinate algebras of compact quantum groups or their homogeneous spaces. We refer to [125], [131] for the story of those algebras, and their notations. Earlier work [56], [61], [129] had revealed the existence of a certain 27-dimensional Hopf algebra closely related to the gauge group of the Standard Model. And the isospectral examples admit quantum symmetries where the labelling parameters are $\lambda = e^{2\pi i\theta}$, complex numbers of modulus 1.

But more conventional 'quantum symmetries' using well-known compact quantum groups such as $\mathrm{SU}_q(2)$ with a *real* parameter $q \neq \pm 1$ seemed to lie outside the framework of spin geometries, as laid out in Chapter 3. Strictly speaking, this is true. However, by mild relaxations of the 'axioms' given above, quantum-homogenous spectral triples have indeed been constructed.

First off the mark were Chakraborty and Pal [25], who worked with the $*$-algebra $\mathcal{O}(\mathrm{SU}_q(2))$, generated by elements $a$ and $b$ satisfying

$$ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b,$$
$$a^*a + q^2b^*b = 1, \quad aa^* + bb^* = 1, \tag{9.3}$$

with $0 < q < 1$. (This convention for the generators is that of [60].) There is a Hilbert space $\mathcal{H}_0 = L^2(\mathrm{SU}_q(2), \psi)$, namely the GNS representation space of the Haar state $\psi$ on the $C^*$-algebra completion $C(\mathrm{SU}_q(2))$, and the GNS or 'regular' representation admits a Peter–Weyl decomposition, just like that of the compact Lie group $\mathrm{SU}(2)$. In [25] a selfadjoint operator $D_0$ is found, such that $(\mathcal{O}(\mathrm{SU}_q(2)), \mathcal{H}_0, D_0)$ is a bona-fide spectral triple, whose metric dimension is 3, as expected.

The symmetry of this spectral triple is described by the Hopf algebra $\mathcal{U}_q(su(2))$, which is a $q$-deformation of the enveloping algebra of the Lie algebra $su(2)$. Its algebra generators are $k, k^{-1}, e, f$ with the commutation relations

$$ek = qke, \quad kf = qfk, \quad k^2 - k^{-2} = (q - q^{-1})(fe - ef),$$

and its coproduct $\Delta$ is given by

$$\Delta k = k \otimes k, \quad \Delta e = e \otimes k + k^{-1} \otimes e, \quad \Delta f = f \otimes k + k^{-1} \otimes f. \tag{9.4}$$

For each $j \in \frac{1}{2}\mathbb{N}$, there is a unique irreducible representation $\sigma_j$ of $\mathcal{U}_q(su(2))$ on a vector space $V_j$ of dimension $2j + 1$. If the vectors $|jm\rangle$, for $m = -j, \ldots, j$, make up an orthonormal basis for $V_j$, then $\sigma_j$ is determined by

$$\sigma_j(k)\,|jm\rangle = q^m\,|jm\rangle, \quad \sigma_j(f)\,|jm\rangle = \sqrt{[j-m]_q[j+m+1]_q}\,|j, m+1\rangle,$$

and $\sigma_j(e) = \sigma_j(f)^*$. The above GNS representation space has the Peter–Weyl decomposition $\mathcal{H}_0 = \bigoplus_{2j=0}^{\infty} V_j \otimes V_j$. We can define an algebra representation $\lambda$ of

$\mathcal{U}_q(su(2))$ on a dense subspace of $\mathcal{H}_0$ by setting $\lambda := \sigma_j \otimes \mathrm{id}$ on each $V_j \otimes V_j$. The construction of [25] asks that $D_0$ commute with this representation. Then $D_0$ is reduced by the finite-dimensional subspaces $V_j \otimes V_j$, and the task is to determine a suitable spectrum of eigenvalues for $D_0$.

This example proved to be a remarkable testing ground for the *local index formula* of Connes and Moscovici [52], whereby the Chern character of a finitely summable spectral triple is expressed as a finite sum of noncommutative integrals, with certain precise coefficients. Although the formula was originally developed for spectral triples arising from foliations (a story we cannot tell here, but see [134] for an excellent review), this quantum-group spectral triple gives all expected terms, with the correct coefficients on the nose [43].

**First-order Dirac operators.** The only drawback to the Chakraborty–Pal example is that $D_0$ is not a 'Dirac operator', because the GNS representation is not a spin representation. A genuine spin geometry was soon found, however. The 'standard' *Podleś sphere* $\mathbb{S}_q^2$ is described by the $*$-subalgebra $\mathcal{O}(\mathbb{S}_q^2) \subset \mathcal{O}(\mathrm{SU}_q(2))$ generated by the quadratic elements $A = b^*b$ and $B = ab^*$; it is also a right comodule for $\mathcal{O}(\mathrm{SU}_q(2))$, so the standard Podleś sphere is an embedded homogeneous space for $\mathrm{SU}_q(2)$. Dąbrowski and Sitarz [62] constructed a spinor representation for this algebra on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, and were able to find operators $D$ and $J$ with the right commutation relations, satisfying the first-order property exactly. Once more, $D$ is invariant under the action of $\mathcal{U}_q(su(2))$, and its eigenvalues are determined (up to a scale factor) by requiring that the first-order property hold. These eigenvalues are the so-called '$q$-integers',

$$\pm [l + \tfrac{1}{2}]_q := \pm \frac{q^{-l-\frac{1}{2}} - q^{l+\frac{1}{2}}}{q^{-1} - q}, \quad \text{for } l = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \dots \tag{9.5}$$

with respective multiplicities $2l + 1$. When $q \to 1$, we recover the Dirac spectrum on the ordinary sphere (2.13).

Krähmer took this construction a step further [135], by showing that such first-order (spin$^c$) Dirac operators can be built on all irreducible quantum flag manifolds: these are $q$-deformations of the Hermitian symmetric spaces $G/K$ where $G$ is a compact semisimple Lie group. In particular, this yields $q$-deformed spheres of any dimension with a first-order Dirac operator.

The metric dimension given by the spectral growth of these Dirac operators is lower than that of the classical case: for $0 < q < 1$ the eigenvalues (9.5) grow exponentially fast, so the metric dimension is 0: see Chapter 3. This 'dimension drop' is a known feature of quantum-group manifolds; for instance, $C(\mathrm{SU}_q(2))$ has only a 1-dimensional circle of characters, and $C(\mathbb{S}_q^2)$ has only one (its 'north pole'). Anticipating this, Connes and Landi proposed [49] that one should look for *isospectral q-deformations*, by borrowing the spectrum from the classical Dirac operator.

**A spin geometry over $SU_q(2)$.** The group manifold $SU(2)$ is a 3-sphere $\mathbb{S}^3 = SO(4)/SO(3) = (SU(2) \times SU(2))/SU(2)$; its full symmetry group is $SU(2) \times SU(2)$, acting on the group manifold by $(p, q) \triangleright x := pxq^{-1}$. In the same way, the full quantum symmetry of the algebra $\mathcal{O}(SU_q(2))$ is given by two commuting actions of the Hopf algebra $\mathcal{U}_q(su(2))$.

When a Hopf algebra $\mathcal{U}$ acts linearly on an algebra $\mathcal{A}$, we require it to satisfy the generalized Leibniz rule: $h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$. Given an algebra representation $\sigma$ of $\mathcal{U}$ on a vector space $V$, we say that a representation $\pi$ of $\mathcal{A}$ on $V$ is $\sigma$-*equivariant* if

$$\sigma(h)\,\pi(x) = \pi(h_{(1)} \triangleright x)\,\sigma(h_{(2)}), \quad \text{for all } h \in \mathcal{U}, \ x \in \mathcal{A}.$$

On the Hilbert space $\mathcal{H}_0 = L^2(SU_q(2), \psi)$, the GNS representation $\pi$ is equivariant under two representations $\lambda$ and $\rho$ of $\mathcal{U}_q(su(2))$ on $\mathcal{H}_0$, for the two respective Hopf actions of $\mathcal{U}_q(su(2))$ on the algebra $\mathcal{O}(SU_q(2))$. They are given on the subspace $V_j \otimes V_j$ by $\lambda := \sigma_j \otimes \mathrm{id}$, as before, and $\rho := \mathrm{id} \otimes \sigma_j$. Using the relations (9.3) and (9.4), one can show [60] that any $(\lambda, \rho)$-equivariant representation of $\mathcal{O}(SU_q(2))$ on $\mathcal{H}_0$ is unitarily equivalent to the GNS representation.

The *spinor space* for $SU_q(2)$ is

$$\mathcal{H} \oplus \mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}^2 = \bigoplus_{2j=0}^{\infty} (V_j \otimes V_j) \otimes V_{\frac{1}{2}} \simeq W_0^{\uparrow} \oplus \bigoplus_{2j \geq 1} W_j^{\uparrow} \oplus W_j^{\downarrow} \qquad (9.6)$$

where $W_0^{\uparrow} = V_{\frac{1}{2}}$, $W_j^{\uparrow} \simeq V_{j+\frac{1}{2}} \otimes V_j$ and $W_j^{\downarrow} \simeq V_{j-\frac{1}{2}} \otimes V_j$, using the Clebsch–Gordan isomorphisms $V_j \otimes V_{\frac{1}{2}} \simeq V_{j+\frac{1}{2}} \oplus V_{j-\frac{1}{2}}$. The *spinor representation* $\pi'$ of $SU_q(2)$ is the unique, up to unitary equivalence, $(\lambda', \rho')$-equivariant representation of $\mathcal{O}(SU_q(2))$ on $\mathcal{H}$, where

$$\lambda'(h) := \lambda(h_{(1)}) \otimes \sigma_{\frac{1}{2}}(h_{(2)}), \quad \rho'(h) := \rho(h) \otimes 1_2 \quad \text{on each } (V_j \otimes V_j) \otimes V_{\frac{1}{2}}.$$

The subspaces $W_j^{\uparrow}$ and $W_j^{\downarrow}$ are joint eigenspaces for $\lambda'(C_q)$ and $\rho'(C_q)$, where $C_q = qk^2 + q^{-1}k^{-2} + (q - q^{-1})^2 ef$ is the Casimir element of $\mathcal{U}_q(su(2))$. The *Dirac operator* $D$ is then defined by requiring that these are also joint eigenspaces for $D$, with respective eigenvalues

$$d_j^{\uparrow} = 2j + \tfrac{3}{2} \ \text{ on } W_j^{\uparrow}, \quad d_j^{\downarrow} = -\left(2j + \tfrac{1}{2}\right) \ \text{ on } W_j^{\downarrow}.$$

These are indeed the eigenvalues for the Dirac operator on the group manifold $SU(2)$ with its invariant metric [115]. Their respective multiplicities are given by $\dim W_j^{\uparrow} = (2j + 1)(2j + 2)$ and $\dim W_j^{\downarrow} = 2j(2j + 1)$. The metric dimension, which is 3, of course, can be computed directly with the Dixmier trace.

It turns out that this spectral triple satisfies the regularity property [204], unlike that of [62]. However, the real structure can only be given approximately. Since we need the relation $JD = DJ$, the conjugation $J$ can be defined on each subspace $W_j^{\uparrow}$ and $W_j^{\downarrow}$ separately. But the antirepresentation $y \mapsto J\pi'(y^*)J^{\dagger}$ does not commute with the representation $\pi'$. On the GNS representation space $\mathcal{H}_0$ there is indeed a Tomita involution $J_0$ such that $J_0\pi(y^*)J_0^{\dagger}$ does commute with each $\pi(x)$, and its double $J_0 \oplus J_0$ on $\mathcal{H}$ will have the commutation property with respect to the spinor representation $\pi'$. However, this double does not preserve the decomposition (9.6) and therefore does not commute with $D$. Thus $J \neq J_0 \oplus J_0$.

Happily, we do not go far wrong: it is shown in [60] that all commutators $[\pi'(x), J\pi'(y^*)J^{\dagger}]$ and all first-order defects $[[D, \pi'(x)], J\pi'(y^*)J^{\dagger}]$, for $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$, are infinitesimals of arbitrarily high order, in the terminology of Chapter 3. Therefore, the spectral triple of [60] may be regarded as a bona-fide spin geometry on the noncommutative space $\mathrm{SU}_q(2)$.

This construction of approximately first-order Dirac operators, along the lines of the $\mathrm{SU}_q(2)$ example, can be transferred to quantum homogeneous spaces. This was done *avant la lettre* for the 'equatorial' Podleś sphere [59]. It also yields nontrivial $K$-cycles on the ordinary Riemann sphere $\mathbb{S}^2$ [200].

## 9.4 Other developments

Space does not permit us to treat all the many new areas that have been opened up to the noncommutative geometric viewpoint. We conclude by giving a few pointers to the literature.

- The survey [42] gives a view of the subject up to 2000, updating the perspective of the book [37]. So many examples of noncommutative spaces are now available that even to catalogue them is a daunting task, but the most comprehensive effort to date is the botanical expedition of Connes and Marcolli [51]. Many of the newer specimens arise from *arithmetic geometry*: see, for example, [44]. The recent book of Marcolli [158] is the most authoritative source for these new encounters of noncommutative geometry and number theory.

- Hopf algebras also appear as carriers of symmetry in the computation of the local index formula for the Chern character [52] of a transversally elliptic operator. To evaluate it, Connes and Moscovici introduced a certain Hopf algebra [53] and developed a theory of *Hopf-cyclic cohomology* to determine its effect on the cyclic cohomology of the coordinate algebra, and thereby compute the index formula. To build this 'Hopf-cyclic cohomology' one needs some extra data that allows one to construct a cyclic operator on the cochains. This construction was

extended to a Hopf-cyclic cohomology with coefficients by Hajac *et al.* [109], which in turn allowed the development of a relative version of this cohomology. For the full story of the development of this purely algebraic theory from its geometric beginnings, see the survey article [54].

- Although noncommutative vector bundles are just finitely generated projective modules, the parallel theory of *noncommutative principal bundles* (and their associated vector bundles) is still under development. The basic concept is well known (namely, a Hopf–Galois extension of algebras [108]), but work remains to be done to understand their role as symmetries of spectral triples. In this regard, we recommend the expository article [7], which explains principal actions and principal-bundle connections in the commutative case from the noncommutative point of view. For examples of principal bundles over isospectral and $q$-deformations, see [110], [140], [143], [144]; in particular, [140] exhibits a Hopf–Galois extension of $\mathcal{O}(\mathrm{SU}_q(2))$.

- The metric distance determined by the Dirac operator may be generalized by replacing $[D, a]$ in (1.8) by $L(a)$, where $L$ is just a Lipschitz seminorm on the algebra $A$. This was suggested by Rieffel [185] as the foundation for a theory of *compact quantum metric spaces*. The development of this concept is surveyed in [186]. For instance, all isospectral deformations of the Connes–Landi type are known to be compact quantum metric spaces [152].

- The *classification of noncommutative 2-tori up to Morita equivalence* was established early on by Rieffel [179]: $A_\theta$ and $A_\phi$ are (strongly) Morita-equivalent if (and only if) $\theta$ and $\phi$ lie in the same orbit of the action of the group $\mathrm{GL}(2, \mathbb{Z})$ on $\mathbb{R}$ by linear fractional transformations. For noncommutative $l$-tori, with $l > 2$, which are Moyal deformations of $C(\mathbb{T}^l)$, the Morita classification was partially found by Rieffel and Schwartz [187] and refined further by Li [151]: the group $\mathrm{SO}(l, l|\mathbb{Z})$ acts by linear fractional transformations on the parameter matrices $\theta \in M_l(\mathbb{R})$ of the deformations, and all elements of the same orbit are strongly Morita equivalent. Other criteria for Morita equivalence of the $C^*$-algebras $A_\theta$ and their smooth subalgebras $\mathcal{A}_\theta$ are given by Elliott and Li [74], [75].

- The Wodzicki residue plays a second role in differential geometry, which lay dormant for a while. On a compact Riemannian manifold of even dimension $n = 2m$, the signature operator $d\delta - \delta d$ has a phase $F = (d\delta - \delta d)(d\delta + \delta d)^{-1}$, defined on the Hilbert space $L^{2,m}(M)$ of middle-dimensional square-integrable forms (minus harmonic forms, if any). This $F$ is a pseudodifferential operator of order zero, so the Wodzicki residue $\mathrm{Wres}([F, f][F, h])$ makes sense for $f, h \in C^\infty(M)$, and is nontrivial if $n > 2$. Connes showed that if $n = 4$, then the formula

$$\mathrm{Wres}([F, f][F, h]) := \int_M f\, P_4(h)\, \Omega$$

determines an elliptic differential operator $P_4$ of order 4 which is conformally invariant, in the sense that a rescaling of the metric gives a compatible rescaling of $P_4$. Ugalde [210] recently showed that this algorithm can produce conformally invariant operators $P_n$ (same formula, with 4 replaced by $n$) for any even dimension $n$, leading to *higher-dimensional conformal invariants*. The case of manifolds with boundary, based on [84], has been treated by Wang [220]. The use of Wodzicki residues to obtain conformal invariants is further explored in [169], [172].

- The theory of spectral triples expounded here is a 'type I' theory, in that all the important operators have discrete spectra, even in the nonunital case. However, building on the theory of generalized singular values [82], one can formulate *type II spectral triples* wherein all relevant operators are affiliated to a fixed, semifinite von Neumann algebra $\mathcal{N}$, Dixmier ideals and traces are defined relative to $\mathcal{N}$, and so forth. The definition is set out in [10], [20] and the Connes–Moscovici local index formula is generalized to the type II case in [21]. The usefulness of this formalism is exemplified by Pask and Rennie in [167], where a type II spectral triple of metric dimension 1 is constructed over a graph $C^*$-algebra.

After a quarter of a century – counting the paper [32] as the conventional starting point – noncommutative geometry is thriving and developing at a rapid pace. This brief list of its recent advances is not exhaustive: there are major areas both in mathematics (for instance, the Baum–Connes conjecture in $K$-theory) and in physics (e.g., the combinatorics of renormalization) that are undergoing rapid progress but which have not rated a mention here. Indeed, the salient feature of noncommutative geometry is its universality: many disparate aspects of mathematics and physics are linked in a unifying theme.

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